

# The Broucke-Hénon orbit and the Schubart Orbit in the planar three-body problem with equal masses

Wentian Kuang

Chern Institute of Mathematics, Nankai University

Tianjin 300071, China

Email: kuangwt1234@163.com

Tiancheng Ouyang

Department of Mathematics, Brigham Young University

Provo, Utah 84602, USA

Email: ouyang@math.byu.edu

Zhifu Xie

Department of Mathematics and Economics

Virginia State University

Petersburg, Virginia 23806, USA

Email: zxie@vsu.edu

Duokui Yan

School of Mathematics and System Science

Beihang University

Beijing 100191, China

Email: duokuiyan@buaa.edu.cn

## Abstract

In this paper, we apply variational method to study the variational properties of two special orbits: the Schubart orbit and the Broucke-Hénon orbit. We show that under an appropriate boundary setting, the action minimizer must be either the Schubart orbit or the Broucke-Hénon orbit.

**Key word:** Variational Method, Schubart Orbit, Broucke-Hénon orbit,  $N$ -body Problem.

**AMS classification number:** 37N05, 70F10, 70F15, 37N30, 70H05, 70F17

## 1 Introduction

Hénon [8, 9] numerically found a one-parameter family of periodic orbits in the planar equal-mass three-body problem, in which the angular momentum is chosen as the non-trivial parameter. In this family, there is a special orbit, as shown in Fig. 1, which has a simple shape and good symmetry

properties. This orbit was also independently discovered by Broucke [3] and it is called as the Broucke-Hénon orbit in this paper.

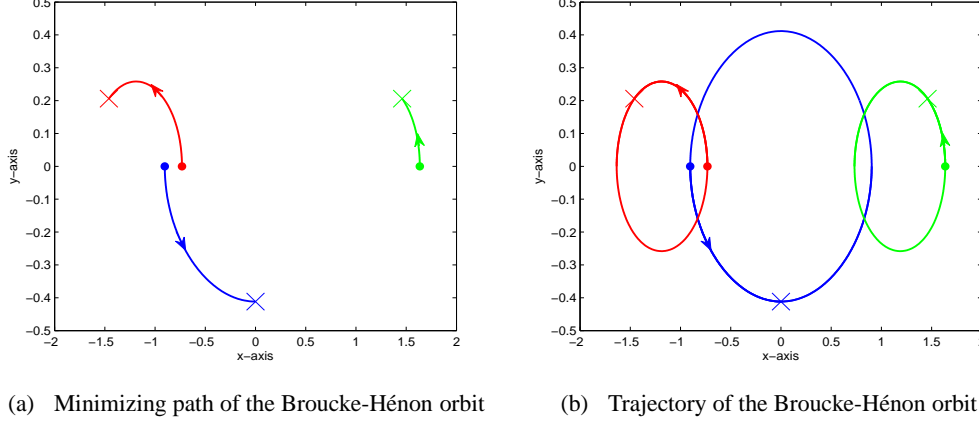


Figure 1: Motion of the Broucke-Hénon orbit. At  $t = 0$ , the three masses (in dots) form a collinear configuration. At  $t = 1$ , they (in crosses) form an isosceles configuration. The path shown in Fig. (a) is an action minimizer of a two-point free boundary value problem connecting a collinear configuration and an isosceles triangle configuration for  $t \in [0, 1]$ . Fig. (b) is the periodic solution extended from the path in Fig. (a).

After Chenciner and Montgomery [1] proved the existence of the figure-eight solution in the planar three-body problem with equal masses by using the variational method, a number of new periodic solutions have been discovered and proven to exist. A workshop on Variational Methods in Celestial Mechanics was organized by Chenciner and Montgomery in 2003 to address the possible applications of variational method in studying the Newtonian N-body problem, while several open problems are proposed by participants. The existence of the Broucke-Hénon orbit is one of these open problems, which was proposed by A. Venturelli. Actually, he noticed that the Schubart orbit with collision [18] (Fig. 2) is on the closure of the homology class  $(1, 0, 1)$  and it is not clear if the Broucke-Hénon orbit (Fig. 1) is a minimizer in the homology class  $(1, 0, 1)$ .

In this paper we use a variational approach to study the existence of the Broucke-Hénon orbit and Schubart orbit. The variational method we used is based on a two-point free boundary value problem with Structural Prescribed Boundary Conditions (SPBC). By using this approach, many stable choreographic solutions [14] and other periodic solutions [15, 25] have been discovered numerically and proved theoretically. In order to obtain the Broucke-Hénon orbit and Schubart orbit by using this variational method with SPBC, we first have to choose the appropriate structural prescribed boundary conditions in the variational frame.

Let  $q_i(t)$  ( $i = 1, 2, \dots, N$ ) denote the position in  $\mathbf{R}^d$  of mass  $m_i > 0$ . Set  $q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \dots \\ q_N(t) \end{bmatrix}$  to be an

$N \times d$  matrix, where  $N$  is the number of bodies and  $d$  is the dimension. Without loss of generality, we

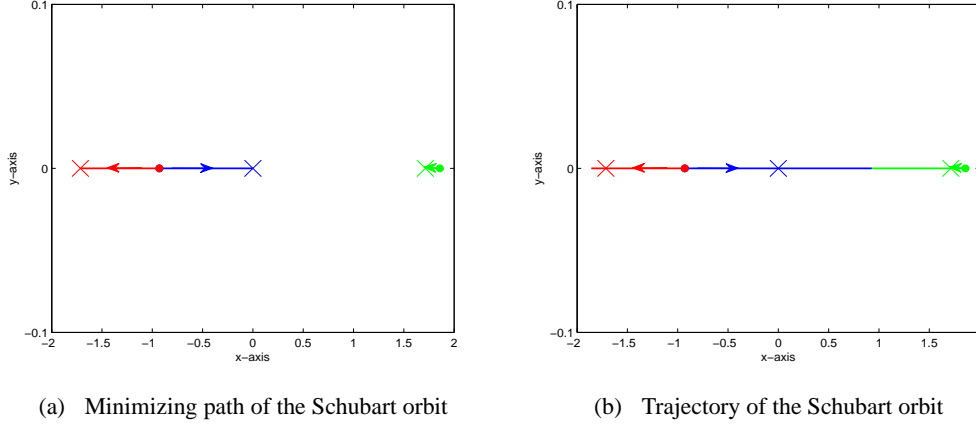


Figure 2: Motion of the Schubart orbit. At  $t = 0$ , body 1 (blue dot) and body 2 (red dot) collide on the  $x$ -axis, while body 3 (green dot) stays away from them. At  $t = 1$ , they (in crosses) form an Euler configuration. The path in Fig. (a) is one forth of the Schubart orbit, which is a two-point free boundary value problem connecting a collinear configuration with a binary collision and an Euler configuration for  $t \in [0, 1]$ . Fig. (b) is the Schubart orbit extended from the path in Fig. (a).

assume that the center of mass is always at the origin. Let

$$\chi = \left\{ q(t) \left| \sum_{i=1}^N m_i q_i(t) = 0 \right. \right\}.$$

The Lagrangian action functional  $\mathcal{A}$  is defined as follows

$$(1) \quad \mathcal{A} = \mathcal{A}(q(t), \dot{q}(t)) = \int_0^1 (K + U) dt,$$

where  $K = K(\dot{q}(t)) = \frac{1}{2} \sum_{i=1}^N m_i |\dot{q}_i(t)|^2$  is the kinetic energy and  $U = U(q(t)) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|}$  is the Newtonian potential. It is known that critical points of the action functional are trajectories that satisfy the equations of motion, i.e. Newton's equations:

$$(2) \quad m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} = \sum_{j=1, j \neq i}^N \frac{m_i m_j (q_j - q_i)}{|q_j - q_i|^3}, \quad 1 \leq i \leq N.$$

Instead of studying a periodic solution directly, a segment of the periodic solution will be considered in a two-point free boundary value problem. The variational method with SPBC is then a two-step minimizing procedure. First, we consider a fixed boundary value problem, which is also known as the Bolza problem. For given boundary matrices  $q(0)$  and  $q(1)$ , there exists an action minimizer  $\mathcal{P}$  connecting them. By Marchal [12] and Chenciner's [2] work, this minimizer  $\mathcal{P}$  is collision free except

the possible collisions at the boundary points. However, if one wants  $\mathcal{P}$  to be part of a periodic solution, the boundaries must be quite special and they should meet certain structural prescribed boundary conditions. Hence, we introduce a second minimizing procedure. Instead of fixing the boundaries, we free several parameters on the boundaries  $q(0)$  and  $q(1)$ . The Lagrangian action functional is then minimized over these parameters. The resulting minimizing path may be extended to a periodic solution or a quasi-periodic solution. There are mainly three challenges to show the existence of such classical solutions in the variational method with SPBC. The first one is the existence of minimizer of the functional under the boundary constraints. The second one is the collision-free of the minimizer on the boundaries. The third one is whether the minimizing path can be extended to a solution that we have expected. With the appropriate choice of SPBC, the three challenges can be resolved.

To introduce the variational method with structural prescribed boundary conditions in detail, we define two boundary matrices  $Qstart$  and  $Qend$  as follows

$$(3) \quad Qstart = \begin{bmatrix} q_1(a_1, \dots, a_k) \\ \dots \\ q_N(a_1, \dots, a_k) \end{bmatrix}, \quad Qend = \begin{bmatrix} q_1(b_1, \dots, b_s) \\ \dots \\ q_N(b_1, \dots, b_s) \end{bmatrix},$$

where  $q_i \in \mathbf{R}^d$  ( $d = 1, 2, 3; i = 1, \dots, N$ ) and  $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_s$  are independent variables. The variational approach is considered as a two-step minimizing process:

$$(4) \quad \inf_{\{(a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}} \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi)\}} \mathcal{A},$$

where  $\mathcal{S}$  is a closed convex subset in  $\mathbf{R}^{k+s}$  and  $\mathcal{A} = \int_0^1 (K + U) dt$ . The first question is the coercivity of the functional  $\mathcal{A}$  in the minimizing problem (4). By making some general assumptions on  $Qstart$  and  $Qend$ , we show that the coercivity of the functional  $\mathcal{A}$  in (4) holds.

**Theorem 1.1.** *Let*

$$(5) \quad Qstart = \begin{bmatrix} q_1(a_1, \dots, a_k) \\ \dots \\ q_N(a_1, \dots, a_k) \end{bmatrix}, \quad Qend = \begin{bmatrix} q_1(b_1, \dots, b_s) \\ \dots \\ q_N(b_1, \dots, b_s) \end{bmatrix},$$

where  $Qstart, Qend \in \chi$ ,  $q_i \in \mathbf{R}^d$  ( $i = 1, \dots, N$ ) and  $a_1, \dots, a_k, b_1, \dots, b_s$  are independent variables. Let  $(a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}$ , where  $\mathcal{S}$  is a closed convex subset in  $\mathbf{R}^{k+s}$ . Assume  $q_i(0) = q_i(a_1, \dots, a_k)$  to be linear with respect to  $a_1, \dots, a_k$ , and  $q_i(1) = q_i(b_1, \dots, b_s)$  to be linear with respect to  $b_1, \dots, b_s$  ( $i = 1, \dots, N$ ). If the intersection of the two configuration subsets is at origin or equal to an empty set, i.e.

$$\{Qstart | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\} \cap \{Qend | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\} = \{\vec{0}\} \text{ or } \emptyset,$$

then there exist a path sequence  $\{\mathcal{P}_{n_l}\}$  and a minimizer  $\mathcal{P}_0$  in  $H^1([0,1], \chi)$ , such that for each  $n_l$ ,

$$\begin{aligned} \mathcal{A}(\mathcal{P}_{n_l}) &= \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi), a_i=a_{i_{n_l}}, b_j=b_{j_{n_l}}, (i=1, \dots, k; j=1, \dots, s)\}} \mathcal{A}(q(t)), \\ \mathcal{A}(\mathcal{P}_0) &= \inf_{\{(a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}} \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi)\}} \mathcal{A} \\ &= \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi), a_i=a_{i_0}, b_j=b_{j_0}, (i=1, \dots, k; j=1, \dots, s)\}} \mathcal{A}. \end{aligned}$$

For  $t \in [0, 1]$ ,  $\mathcal{P}_{n_l}(t)$  converges to  $\mathcal{P}_0(t)$  uniformly. In particular,

$$\lim_{n_l \rightarrow \infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l \rightarrow \infty} b_{j_{n_l}} = b_{j_0}, \quad i = 1, \dots, k; j = 1, \dots, s.$$

As an extension of the above theorem, the following result holds:

**Lemma 1.2.** *Let*

$$(6) \quad Qstart = \begin{bmatrix} q_1(a_1, \dots, a_k) \\ \dots \\ q_N(a_1, \dots, a_k) \end{bmatrix}, \quad Qend = \begin{bmatrix} q_1(b_1, \dots, b_s) \\ \dots \\ q_N(b_1, \dots, b_s) \end{bmatrix} R(\theta),$$

where  $Qstart, Qend \in \chi$ ,  $R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ ,  $q_i \in \mathbf{R}^2$ , ( $i = 1, \dots, N$ ) and  $a_1, \dots, a_k, b_1, \dots, b_s, \theta$  are independent variables. Let  $(a_1, \dots, a_k, b_1, \dots, b_s, \theta) \in \mathcal{S}$ , where  $\mathcal{S}$  is a closed convex subset in  $\mathbf{R}^{k+s+1}$ . Assume  $q_i(a_1, \dots, a_k)$  to be linear with respect to  $a_1, \dots, a_k$ , and  $q_i(b_1, \dots, b_s)$  to be linear with respect to  $b_1, \dots, b_s$  ( $i = 1, \dots, N$ ). If the intersection of the two configuration subsets is at origin or equal to an empty set, i.e.

$$\{Qstart | (a_1, \dots, a_k, b_1, \dots, b_s, \theta) \in \mathcal{S}\} \cap \{Qend | (a_1, \dots, a_k, b_1, \dots, b_s, \theta) \in \mathcal{S}\} = \{\vec{0}\} \text{ or } \emptyset,$$

then there exist a path sequence  $\{\mathcal{P}_{n_l}\}$  and a minimizer  $\mathcal{P}_0$  in  $H^1([0, 1], \chi)$ , such that for each  $n_l$ ,

$$\begin{aligned} \mathcal{A}(\mathcal{P}_{n_l}) &= \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0, 1], \chi), a_i=a_{i_{n_l}}, b_j=b_{j_{n_l}}, (i=1, \dots, k; j=1, \dots, s), \theta=\theta_{n_l}\}} \mathcal{A}(q(t)), \\ \mathcal{A}(\mathcal{P}_0) &= \inf_{\{(a_1, \dots, a_k, b_1, \dots, b_s, \theta) \in \mathcal{S}\}} \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0, 1], \chi)\}} \mathcal{A} \\ &= \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0, 1], \chi), a_i=a_{i_0}, b_j=b_{j_0}, (i=1, \dots, k; j=1, \dots, s), \theta=\theta_0\}} \mathcal{A}. \end{aligned}$$

For  $t \in [0, 1]$ ,  $\mathcal{P}_{n_l}(t)$  converges to  $\mathcal{P}_0(t)$  uniformly. In particular,

$$\lim_{n_l \rightarrow \infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l \rightarrow \infty} b_{j_{n_l}} = b_{j_0}, \quad (i = 1, \dots, k; j = 1, \dots, s), \quad \lim_{n_l \rightarrow \infty} \theta_{n_l} = \theta_0.$$

To study the Broucke-Hénon orbit, we concentrate on the following minimizing problem

$$(7) \quad \mathcal{A}(\mathcal{P}) = \inf_{\{(a_1, a_2, b_1, b_2) \in \mathcal{S}_1\}} \inf_{\{q(0)=Qstart_1, q(1)=Qend_1, q(t) \in H^1([0, 1], \chi)\}} \mathcal{A},$$

where  $\mathcal{A} = \int_0^1 (K + U) dt$  is defined by (1),

$$(8) \quad Qstart_1 = \begin{bmatrix} -2a_1 - a_2 & 0 \\ a_1 - a_2 & 0 \\ a_1 + 2a_2 & 0 \end{bmatrix}, \quad Qend_1 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and  $(a_1, a_2, b_1, b_2) \in \mathcal{S}_1$ , with

$$\mathcal{S}_1 = \{a_1 \geq 0, \quad a_2 \geq 0, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}\}.$$

Based on the works of Chenciner [2] and Marchal [12], the challenge of showing the existence of the Broucke-Hénon orbit is to exclude possible boundary collisions and to find the extension properties. Our main result is as follows, while its proof can be found in Theorem 5.4 and Lemma 6.2.

**Theorem 1.3.** *The minimizer  $\mathcal{P}$  in (7) is either the Schubart orbit or the Broucke-Hénon orbit.*

The paper is organized as follows. Section 2 introduces a general result of coercivity. Section 3 excludes the total collision in the minimizer  $\mathcal{P}$  of (7). Section 4 excludes possible binary collisions of  $\mathcal{P}$  at  $t = 1$  and studies the behavior of binary collisions in  $\mathcal{P}$  at  $t = 0$ . Section 5 shows that  $\mathcal{P}$  must coincide with the Schubart orbit in Fig. 2 whenever  $\mathcal{P}$  has collision singularities. In the last section, we show that if  $\mathcal{P}$  has no collision, it can be extended to a periodic orbit with  $D_2$  symmetry.

## 2 Coercivity under general boundary settings

In this section, we prove Theorem 1.1 and Lemma 1.2 of the coercivity of the Lagrangian action functional under the structural prescribed boundary conditions in the N-body problem. Note that  $L = K + U \geq 0$ , hence there exists some  $M_0 \geq 0$ , such that

$$\inf_{\{(a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}} \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi)\}} \mathcal{A} = M_0.$$

The proof follows by the Arzelà-Ascoli theorem. Basically, we can find a sequence  $\mathcal{P}_n$ , such that the action of the sequence  $\mathcal{A}(\mathcal{P}_n)$  approaches  $M_0$ . Then we show the uniform boundedness and equicontinuity of the sequence. Hence, by the Arzelà-Ascoli theorem, there is a subsequence  $\mathcal{P}_{n_l}$  which converges uniformly to a minimizer  $\mathcal{P}_0$ .

Note that there exist sequences  $a_{i_n}$  and  $b_{j_n}$ , such that the minimum action value  $M_0$  can be reached by a path sequence  $\mathcal{P}_n \in H^1([0,1], \chi)$ , which satisfies

$$\mathcal{A}(\mathcal{P}_n) = \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1], \chi), a_i=a_{i_n}, b_j=b_{j_n}, (i=1, \dots, k; j=1, \dots, s)\}} \mathcal{A},$$

and  $\mathcal{A}(\mathcal{P}_n) \in [M_0, M_0 + \frac{1}{2^n}]$ . It is clear that  $\mathcal{A}(\mathcal{P}_n) \in [M_0, M_0 + 1]$  for all  $n$ . Next, we show the path sequence  $\{\mathcal{P}_n\}$  is uniformly bounded.

We rewrite  $Qstart$  and  $Qend$  as  $dN \times 1$  vectors:

$$\widetilde{Qstart} = \begin{bmatrix} q_1^T(a_1, \dots, a_k) \\ \dots \\ q_N^T(a_1, \dots, a_k) \end{bmatrix}, \quad \widetilde{Qend} = \begin{bmatrix} q_1^T(b_1, \dots, b_s) \\ \dots \\ q_N^T(b_1, \dots, b_s) \end{bmatrix}.$$

Since  $a_1, \dots, a_k, b_1, \dots, b_s$  are all independent and  $(a_1, \dots, a_k, b_1, \dots, b_s)$  is in a closed convex subset  $\mathcal{S}$ , it follows that  $\{Qstart | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}$  and  $\{Qend | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}$  are closed. By assumption, the two linear closed subsets satisfy

$$\{Qstart | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\} \cap \{Qend | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\} = \{\vec{0}\} \text{ or } \emptyset.$$

It follows that  $\{\widetilde{Qstart} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}$  is a closed convex subset of a  $k$ -dimensional linear space  $U_k$  and  $\{\widetilde{Qend} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}$  is a closed convex subset of a  $s$ -dimensional linear

space  $V_s$ . Let  $\{u_1, \dots, u_k\}$  be an orthonormal basis of  $U_k$  and  $\{v_1, \dots, v_s\}$  be an orthonormal basis of  $V_s$ . Choose nonzero vectors

$$\vec{u} \in \left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}, \quad \vec{v} \in \left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\},$$

such that  $k_1 \vec{u} \in \left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$  and  $k_1 \vec{v} \in \left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$  for all big enough  $k_1 > 0$ . There exist constants  $g_i, h_j$  ( $1 \leq i \leq k, 1 \leq j \leq s$ ), such that

$$\begin{aligned} \frac{\vec{u}}{\|\vec{u}\|} &= g_1 u_1 + \dots + g_k u_k, & \sum_{i=1}^k g_i^2 &= 1, \\ \frac{\vec{v}}{\|\vec{v}\|} &= h_1 v_1 + \dots + h_s v_s, & \sum_{j=1}^s h_j^2 &= 1. \end{aligned}$$

Note that  $g_i$  and  $h_j$  ( $1 \leq i \leq k, 1 \leq j \leq s$ ) satisfy  $\sum_{i=1}^k g_i^2 = \sum_{j=1}^s h_j^2 = 1$ . So they are on a compact set.

Since  $\left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$  and  $\left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$  both are closed, it follows that the inner product of  $\frac{\vec{u}}{\|\vec{u}\|}$  and  $\frac{\vec{v}}{\|\vec{v}\|}$ :

$$\left\langle \frac{\vec{u}}{\|\vec{u}\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\rangle = \sum_{1 \leq i \leq k, 1 \leq j \leq s} g_i h_j \langle u_i, v_j \rangle = \cos(\vec{u}, \vec{v})$$

can reach its maximum  $K_0$ . If  $K_0 = 1$ , there exist two vectors  $\vec{u} \in \left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$  and  $\vec{v} \in \left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$ , such that they share the same direction. Note that  $\mathcal{S}$  is convex, it follows that there exists some  $k_1 > 0$ , such that

$$k_1 \vec{u} \in \left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\} \cap \left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}.$$

Contradiction! Hence,  $K_0 < 1$ . It implies that for unbounded direction

$$\vec{u} \in \left\{ \widetilde{Qstart} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}, \quad \vec{v} \in \left\{ \widetilde{Qend} \mid (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\},$$

the angle between them is less than  $\pi$ .

On the other hand,  $\mathcal{A}(\mathcal{P}_n) \leq M_0 + 1$ . If  $0 \leq t_1 < t_2 \leq 1$ , we have

$$\frac{m_j \left\| q_j^n(t_2) - q_j^n(t_1) \right\|^2}{2d(t_2 - t_1)} \leq \int_{t_1}^{t_2} \frac{m_j \left\| \dot{q}_j(t) \right\|^2}{2} dt \leq \mathcal{A}(\mathcal{P}_n) \leq M_0 + 1.$$

It implies that for any  $1 \leq j \leq N$  and any  $t_1, t_2$  satisfying  $0 \leq t_1 < t_2 \leq 1$ ,

$$(9) \quad \left\| q_j^n(t_2) - q_j^n(t_1) \right\| \leq \sqrt{\frac{2d(t_2 - t_1)(M_0 + 1)}{m_j}} \leq \sqrt{\frac{6(M_0 + 1)}{m_j}}.$$

Let  $m^* = \min\{m_1, m_2, \dots, m_N\}$ . Then for all  $1 \leq j \leq N$ ,

$$\|q_j^n(t_2) - q_j^n(t_1)\| \leq \sqrt{\frac{6(M_0 + 1)}{m^*}}.$$

In each  $\mathcal{P}_n$ , its element  $q^{(n)}(t) = \begin{bmatrix} q_1^{(n)}(a_1, \dots, a_k) \\ \dots \\ q_N^{(n)}(a_1, \dots, a_k) \end{bmatrix}$  can be rewritten as  $\tilde{q}^{(n)}(t) = \begin{bmatrix} q_1^{(n)}(a_1, \dots, a_k)^T \\ \dots \\ q_N^{(n)}(a_1, \dots, a_k)^T \end{bmatrix}$ .

Then for any  $t \in [0, 1]$ ,

$$(10) \quad \|\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(t)\| \leq N \sqrt{\frac{6(M_0 + 1)}{m^*}}.$$

The uniform boundedness is discussed in two cases. If both  $\frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|}$  and  $\frac{\tilde{q}^{(n)}(1)}{\|\tilde{q}^{(n)}(1)\|}$  are unbounded directions in the corresponding boundary configuration sets, the convexity condition implies that the two vectors satisfy

$$k_1 \frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|} \in \left\{ \widetilde{Qstart} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$$

$$k_1 \frac{\tilde{q}^{(n)}(1)}{\|\tilde{q}^{(n)}(1)\|} \in \left\{ \widetilde{Qend} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S} \right\}$$

for big enough  $k_1 > 0$ . It follows that

$$\begin{aligned} \frac{6N^2(M_0 + 1)}{m^*} &\geq \|\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(1)\|^2 \\ &= \|\tilde{q}^{(n)}(0)\|^2 + \|\tilde{q}^{(n)}(1)\|^2 - 2\|\tilde{q}^{(n)}(0)\|\|\tilde{q}^{(n)}(1)\|\cos(\tilde{q}^{(n)}(0), \tilde{q}^{(n)}(1)) \\ &\geq \|\tilde{q}^{(n)}(0)\|^2 + \|\tilde{q}^{(n)}(1)\|^2 - 2K_0\|\tilde{q}^{(n)}(0)\|\|\tilde{q}^{(n)}(1)\| \\ &= \left[ K_0\|\tilde{q}^{(n)}(0)\| - \|\tilde{q}^{(n)}(1)\| \right]^2 + (1 - K_0^2)\|\tilde{q}^{(n)}(0)\|^2 \\ &\geq (1 - K_0^2)\|\tilde{q}^{(n)}(0)\|^2. \end{aligned}$$

Hence

$$(11) \quad \|\tilde{q}^{(n)}(0)\| \leq \sqrt{\frac{6N^2(M_0 + 1)}{m^*(1 - K_0^2)}}.$$

By inequalities (10) and (11), it follows that for any  $t \in [0, 1]$ ,

$$(12) \quad \|\tilde{q}^{(n)}(t)\| \leq \|\tilde{q}^{(n)}(0) - \tilde{q}^{(n)}(t)\| + \|\tilde{q}^{(n)}(0)\| \leq N \sqrt{\frac{6(M_0 + 1)}{m^*}} + N \sqrt{\frac{6(M_0 + 1)}{m^*(1 - K_0^2)}},$$

which is a uniform bound for  $\|\tilde{q}^{(n)}(t)\|$ .



The other case is that one of the directions  $\frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|}$  or  $\frac{\tilde{q}^{(n)}(1)}{\|\tilde{q}^{(n)}(1)\|}$  is a bounded direction in its corresponding configuration subset. Without loss of generality, we assume  $\frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|}$  is the bounded direction in  $\{\widetilde{Qstart} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}$ . That is, there exists some  $k_2 > 0$ , such that

$$k_2 \frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|} \in \{\widetilde{Qstart} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\},$$

$$k_3 \frac{\tilde{q}^{(n)}(0)}{\|\tilde{q}^{(n)}(0)\|} \notin \{\widetilde{Qstart} | (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}\}, \text{ if } k_3 > k_2.$$

It implies that  $\tilde{q}^{(n)}(0)$  is uniformly bounded by  $k_2$ . By inequality (10),  $\tilde{q}^{(n)}(t)$  is uniformly bounded. Therefore, the path sequence  $\mathcal{P}_n = \mathcal{P}_n(t)$  is uniformly bounded.

Next, we show the path sequence  $\{\mathcal{P}_n = \mathcal{P}_n(t)\}$  is equi-continuous. In fact, by inequality (9),

$$\|q_j^n(t_2) - q_j^n(t_1)\| \leq \sqrt{\frac{6(M_0 + 1)}{m^*}} |t_2 - t_1|^{1/2}.$$

Then for any  $\varepsilon > 0$ , let  $\delta = \frac{\varepsilon^2 m^*}{6(M_0 + 1)}$ . Whenever  $|t_2 - t_1| \leq \delta$ , the following inequality holds:

$$\|q_j^n(t_2) - q_j^n(t_1)\| \leq \sqrt{\frac{6(M_0 + 1)}{m^*}} |t_2 - t_1|^{1/2} = \varepsilon.$$

It implies that for each  $j \in [1, N]$ ,  $q_j^n(t)$  is equi-continuous. It follows that the path sequence  $\mathcal{P}_n = \mathcal{P}_n(q(t))$  is equi-continuous.

By the Arzelà-Ascoli theorem, there exists a subsequence  $\{\mathcal{P}_{n_l}\}$  which converges uniformly. The limit  $\mathcal{P}_0 = \mathcal{P}_0(q(t))$  is in  $H^1([0, 1], \mathcal{X})$  and it satisfies

$$\lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(t) = \mathcal{P}_0(t), \quad \text{for all } t \in [0, 1].$$

In particular,

$$\lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(0) = \mathcal{P}_0(0), \quad \lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(1) = \mathcal{P}_0(1).$$

It follows that

$$\lim_{n_l \rightarrow \infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l \rightarrow \infty} b_{j_{n_l}} = b_{j_0}, \quad i = 1, \dots, k; j = 1, \dots, s.$$

And  $\mathcal{P}_0$  satisfies

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0, 1], \mathcal{X}), a_i = a_{i_0}, b_j = b_{j_0}, (i=1, \dots, k; j=1, \dots, s)\}} \mathcal{A}.$$

The proof is complete.

**Proof of Lemma 1.2:** Note that the new form  $Qend$  is not linear with respect to  $\theta$ . However,  $R(\theta)$  has a period  $2\pi$ . In other words, it is equivalent to consider  $\theta \in S_\theta \subset [0, 2\pi]$ , where  $S_\theta$  is closed. The

proof of this lemma also follows by the Arzelà-Ascoli theorem. For each given  $\theta$ , by Theorem 1.1, there exists a path  $\mathcal{P}_\theta$  which satisfies

$$\mathcal{A}(\mathcal{P}_\theta) = \inf_{\{(a_1, \dots, a_k, b_1, \dots, b_s) \in S\}} \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1]; \chi)\}} \mathcal{A}.$$

For  $\theta \in S_\theta \subset [0, 2\pi]$ , the infimum value  $\inf_{\theta \in S_\theta} \mathcal{A}(\mathcal{P}_\theta) = M_\theta$  exists. Then one can take a sequence of paths  $\mathcal{P}_{\theta_n}$ , such that

$$\lim_{n \rightarrow \infty} \mathcal{A}(\mathcal{P}_{\theta_n}) = M_\theta.$$

It is clear that the equi-continuous condition holds for  $\mathcal{P}_{\theta_n}$ . By inequity (12), the uniform bound of  $\|\mathcal{P}_{\theta_n}\|$  is a continuous function of  $\theta$ . It follow that for  $\theta \in S_\theta \subset [0, 2\pi]$ , this uniform bound has a maximum value which is independent of  $\theta$ . Hence, the uniform boundedness of  $\mathcal{P}_{\theta_n}$  is satisfied. Therefore, by the Arzelà-Ascoli theorem, there exists a subsequence  $\mathcal{P}_{\theta_{n_l}}$  which converges uniformly. The limit  $\mathcal{P}_0 = \mathcal{P}_0(q(t))$  is in  $H^1([0, 1], \chi)$  and it satisfies

$$\lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(t) = \mathcal{P}_0(t), \quad \text{for all } t \in [0, 1].$$

In particular,

$$\lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(0) = \mathcal{P}_0(0), \quad \lim_{n_l \rightarrow \infty} \mathcal{P}_{n_l}(1) = \mathcal{P}_0(1).$$

It follows that

$$\lim_{n_l \rightarrow \infty} a_{i_{n_l}} = a_{i_0}, \quad \lim_{n_l \rightarrow \infty} b_{j_{n_l}} = b_{j_0}, \quad i = 1, \dots, k; j = 1, \dots, s.$$

And  $\mathcal{P}_0$  satisfies

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{q(0)=Qstart, q(1)=Qend, q(t) \in H^1([0,1]; \chi), a_i=a_{i_0}, b_j=b_{j_0}, (i=1, \dots, k; j=1, \dots, s), \theta=\theta_0\}} \mathcal{A}.$$

The proof is complete.

*Remark 2.1.* Note that we do not require the subsets  $Qstart$  and  $Qend$  to be linear subspaces. In fact, if the domain of the free parameters  $a_i, b_j (i = 1, \dots, k; j = 1, \dots, s)$  is  $\mathbb{R}^{k+s}$ , then  $Qstart$  and  $Qend$  are two linear subspaces. In such cases, one can also apply the symmetry constraint and topological constraint methods to show their existence, such as in [1, 7, 14, 15] and references therein. However, there are orbits in which one of the free boundaries  $Qstart$  or  $Qend$  has to be restricted in a half space or in a closed convex subset, such as in [5]. In these cases, our argument is helpful in showing the existence and finding the exact variational property of the periodic orbits. The Broucke-Hénon orbit in our paper is an example of such variational problems with constraints on boundaries.

### 3 Exclusion of total collision

In what follows, we concentrate on the minimizing problem

$$(13) \quad \inf_{\{(a_1, a_2, b_1, b_2) \in S_1\}} \inf_{\{q(0)=Qstart_1, q(1)=Qend_1, q(t) \in H^1([0,1]; \chi)\}} \mathcal{A},$$

where  $\mathcal{A} = \int_0^1 (K + U) dt$  is defined by (1),

$$(14) \quad Q_{start_1} = \begin{bmatrix} -2a_1 - a_2 & 0 \\ a_1 - a_2 & 0 \\ a_1 + 2a_2 & 0 \end{bmatrix}, \quad Q_{end_1} = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and  $(a_1, a_2, b_1, b_2) \in S_1$ , with

$$(15) \quad S_1 = \{a_1 \geq 0, \quad a_2 \geq 0, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}\}.$$

By Theorem 1.1, there exists an action minimizer  $\mathcal{P}$  of the minimizing problem (13). Based on the celebrating works of Chenciner [2] and Marchal [12], the main challenge in the existence proof is to exclude possible boundary collisions.

Let  $q_i = (q_{ix}, q_{iy})$  ( $i = 1, 2, 3$ ). From the definition of  $Q_{start_1}$ ,  $Q_{end_1}$  in (14) and  $S$  in (15), it implies that at  $t = 0$ , the three bodies are on the  $x$ -axis and satisfy the order  $q_{1x}(0) \leq q_{2x}(0) \leq q_{3x}(0)$ . The possible collisions at  $t = 0$  are binary collision between bodies 1 and 2, binary collision between bodies 2 and 3 and total collision. The possible collisions at  $t = 1$  are binary collisions between bodies 1 and 2, 1 and 3, 2 and 3 and total collision. In this section, we define a testing path to exclude the possibility of total collisions on both boundaries.

We first find a lower bound of actions for paths with total collision. We consider the collinear Kepler problem:  $\frac{d^2\gamma}{dt^2} = -\lambda\gamma^{-2}$ , where  $\lambda > 0$  is a constant. There is a unique solution such that  $\gamma_{\lambda,1}(0) = \gamma_{\lambda,1}(1) = 0$  and  $\dot{\gamma}_{\lambda,1}(t) > 0$  for any  $t \in (0, T_0)$ . This is a degenerate Kepler motion with period 2. It is known [10] that  $\gamma_{\lambda,1}$  minimizes the action functional for the collinear Kepler motion

$$\int_0^1 \frac{1}{2} \dot{\gamma}^2 + \frac{\lambda}{\gamma} dt$$

on  $\{\gamma \in H^1([0, 1], \mathbb{R}) | \gamma(t) = 0 \text{ for some } t \in [0, 1]\}$ . And the minimum action value is

$$\frac{3}{2} \pi^{2/3} \lambda^{2/3}.$$

Note that the central configuration of three equal masses can be either a collinear configuration or an equilateral triangle configuration. In the case of collinear configuration, let  $q_1(t) = \gamma(t)$ ,  $q_2(t) = 0$  and  $q_3(t) = -\gamma(t)$ . It follows that

$$\mathcal{A}_{collinear} = \int_0^1 (K + U) dt = \int_0^1 \dot{\gamma}^2 + \frac{5}{2\gamma} dt = 2 \int_0^1 \frac{1}{2} \dot{\gamma}^2 + \frac{5}{4\gamma} dt.$$

It implies that

$$(16) \quad \mathcal{A}_{collinear} \geq 2 \frac{3}{2} \pi^{2/3} \left( \frac{5}{4} \right)^{2/3} \approx 7.4672.$$

In the case of equilateral triangle configuration, we can set  $q_1(t) = \gamma(t)$ ,  $q_2(t) = \gamma(t)R(2\pi/3)$  and  $q_3(t) = \gamma(t)R(-2\pi/3)$ . It follows that

$$\mathcal{A}_{triangle} = \int_0^1 (K + U) dt = \int_0^1 \frac{3}{2} \dot{\gamma}^2 + \frac{3}{\sqrt{3}\gamma} dt = 3 \int_0^1 \frac{1}{2} \dot{\gamma}^2 + \frac{1}{\sqrt{3}\gamma} dt.$$

It implies that

$$(17) \quad \mathcal{A}_{triangle} \geq 3\frac{3}{2}\pi^{\frac{2}{3}}\left(\frac{1}{\sqrt{3}}\right)^{\frac{2}{3}} \approx 6.6927.$$

Inequalities (16) and (17) imply that the lower bound of action for paths with total collision is about 6.6927. Therefore, if we can find a testing path  $\mathcal{P}_{test} \in H^1([0, 1], \chi)$  with action less than 6.69, then there is no total collision in the minimizer  $\mathcal{P}_0$  of (13).

By [22], a piece of Schubart orbit can be characterized as an action minimizer between two collinear configurations:

$$Qstart_2 = \begin{bmatrix} -c_1 \\ -c_1 \\ 2c_1 \end{bmatrix}, \quad Qend_2 = \begin{bmatrix} 0 \\ -d_1 \\ d_1 \end{bmatrix},$$

with  $(c_1, d_1) \in \mathbb{R}^2$ . Our testing path is taken as a 1-dimensional path connecting  $Qstart_2$  and  $Qend_2$ , which can be seen as an approximation of the Schubart orbit. The definition of the testing path  $\hat{q}(t) = \begin{bmatrix} \hat{q}_{1x}(t) & 0 \\ \hat{q}_{2x}(t) & 0 \\ \hat{q}_{3x}(t) & 0 \end{bmatrix}$  is as follows.

$$(18) \quad \hat{q}_{1x}(t) = \begin{cases} -\frac{9}{10} + t^{\frac{2}{3}} & 0 \leq t \leq \frac{1}{8}, \\ \frac{26}{35}t - \frac{26}{35} & \frac{1}{8} \leq t \leq 1, \end{cases}$$

$$(19) \quad \hat{q}_{2x}(t) = \begin{cases} -\frac{9}{10} - t^{\frac{2}{3}} & 0 \leq t \leq \frac{1}{8}, \\ -\frac{26}{35}t - \frac{37}{35} & \frac{1}{8} \leq t \leq 1, \end{cases}$$

and  $\hat{q}_{3x}(t) \equiv \frac{9}{5}$ . The path  $\hat{q} \in H^1([0, 1], \chi)$  and it satisfies the boundary conditions

$$\hat{q}(0) \in \{Qstart_1 | (a_1, a_2, b_1, b_2) \in \mathcal{S}_1\}, \quad \hat{q}(1) \in \{Qend_1 | (a_1, a_2, b_1, b_2) \in \mathcal{S}_1\}.$$

**Lemma 3.1.** Assume  $\hat{q}(t) \in H^1([0, 1], \chi)$  is defined by (18) and (19), then

$$\mathcal{A}(\hat{q}(t)) < 3.5383.$$

*Proof.* The action of  $\hat{q} \in H^1([0, 1], \chi)$  is calculated in two parts. When  $t \in [0, 1/8]$ , the action of  $\hat{q}(t)$  is

$$(20) \quad \begin{aligned} \mathcal{A}_1 &= \int_0^{1/8} L(\dot{\hat{q}}, \hat{q}) dt \\ &= \int_0^{0.125} \frac{4}{9} t^{-2/3} + \frac{1}{2t^{2/3}} + \frac{1}{2.7 - t^{2/3}} + \frac{1}{2.7 + t^{2/3}} dt \\ &< \frac{3}{2} \left( \frac{4}{9} + \frac{1}{2} \right) + 0.125 \times \frac{5.4}{2.7^2 - (0.125)^{4/3}} \\ &\approx 1.5100. \end{aligned}$$

When  $t \in [1/8, 1]$ , the action of  $\hat{q}(t)$  is

$$\begin{aligned}
(21) \quad & \mathcal{A}_2 \\
&= \int_{1/8}^1 L(\dot{\hat{q}}, \hat{q}) dt \\
&= \frac{7}{8} \left( \frac{26}{35} \right)^2 + \int_{0.125}^1 \frac{35}{52t+11} + \frac{35}{26t+100} + \frac{35}{89-26t} dt \\
&= \frac{7}{8} \left( \frac{26}{35} \right)^2 + \frac{35}{52} \ln(52t+11) \Big|_{0.125}^1 + \frac{35}{26} \ln(26t+100) \Big|_{0.125}^1 - \frac{35}{26} \ln(89-26t) \Big|_{0.125}^1 \\
&\approx 2.0281.
\end{aligned}$$

Therefore, by equations (20) and (21), the action of  $\hat{q}(t)$  ( $t \in [0, 1]$ ) satisfies

$$\mathcal{A}(\hat{q}(t)) = \mathcal{A}_1 + \mathcal{A}_2 < 1.5101 + 2.0282 = 3.5383.$$

The proof is complete.  $\square$

**Corollary 3.2.** *The minimizing path  $\mathcal{P}$  has no triple collision and*

$$\mathcal{A}(\mathcal{P}) < 3.5383.$$

*Proof.* The definition of  $\hat{q}(t)$  ( $t \in [0, 1]$ ) implies that  $\hat{q}(t) \in H^1([0, 1], \chi)$  and

$$\begin{aligned}
\hat{q}(0) &= \begin{bmatrix} -0.9 & 0 \\ -0.9 & 0 \\ 1.8 & 0 \end{bmatrix} \in \{Qstart_1 \mid \{a_1, a_2, b_1, b_2\} \in \mathcal{S}_1\}, \\
\hat{q}(1) &= \begin{bmatrix} 0 & 0 \\ -1.8 & 0 \\ 1.8 & 0 \end{bmatrix} \in \{Qend_1 \mid \{a_1, a_2, b_1, b_2\} \in \mathcal{S}_1\},
\end{aligned}$$

where  $Qstart_1$  and  $Qend_1$  are defined by (14) and  $\mathcal{S}_1$  is defined by (15). By Lemma 3.1, the action of the testing path  $\hat{q}(t) \in H^1([0, 1], \chi)$  is strictly less than 3.5383. It follows that

$$\mathcal{A}(\mathcal{P}) < 3.5383.$$

By equations (17) and (16), the lower bound of actions of paths containing total collision is 6.6927, which is strictly greater than 3.5383. Hence, the action minimizer  $\mathcal{P} \in H^1([0, 1], \chi)$  has no total collision. The proof is complete.  $\square$

Let a Schubart orbit have period 4. At  $t = 0$ , we assume this orbit starts with a binary collision between bodies 1 and 2, and at  $t = 1$ , let body 1 be at 0 and bodies 2 and 3 be symmetrically located on the two sides. It is clear that  $\hat{q}(t)$  ( $t \in [0, 1]$ ) is always on the  $x$ -axis and its boundaries satisfy  $\hat{q}(0) \in Qstart_2$  and  $\hat{q}(1) \in Qend_2$ , where

$$Qstart_2 = \begin{bmatrix} -c_1 \\ -c_1 \\ 2c_1 \end{bmatrix}, \quad Qend_2 = \begin{bmatrix} 0 \\ -d_1 \\ d_1 \end{bmatrix},$$

with  $(c_1, d_1) \in \mathbb{R}^2$ . According to the variational property of the Schubart orbit in [22], the action of the testing path  $\hat{q}(t)$  ( $t \in [0, 1]$ ) is greater than the action of the Schubart orbit in  $[0, 1]$ . Hence, the following corollary holds.

**Corollary 3.3.** *Let the Schubart orbit have period 4 and denote its action in one period ( $t \in [0, 4]$ ) by  $\mathcal{A}_{Schubart}$ . Then*

$$\frac{1}{4} \mathcal{A}_{Schubart} < 3.5383.$$

## 4 Revisit the Kepler theorem

In this section, we apply the Lambert theorem to study the possible binary collision in the minimizer  $\mathcal{P}_0 \in H^1([0, 1], \chi)$ . Note that at  $t = 0$ , the boundary  $Qstart_1$  has order restriction  $q_{1x}(0) \leq q_{2x}(0) \leq q_{3x}(0)$ . At  $t = 1$ ,  $\{Qend_1 | b_1, b_2 \in \mathbb{R}\}$  is a two dimensional vector space. A standard local deformation argument can be applied to show that  $Qend_1$  contains no binary collision. However, our result implies that it is impossible to exclude binary collisions at  $t = 0$  by local deformation.

We consider a one-end free boundary value problem in the kepler problem. Let  $q_i = (q_{ix}, q_{iy})$  be the position of mass  $m_i$  ( $i = 1, 2$ ). Let  $\mathbf{r}(t) = q_1(t) - q_2(t)$ ,  $\alpha = m_1 m_2$  and  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ . Assume the free-end is at  $t = 0$ . Set the two masses  $m_1$  and  $m_2$  to be on the  $x$ -axis with a given order ( $q_{1x}(0) \leq q_{2x}(0)$ ) at  $t = 0$ . When  $q_1(0) = q_2(0)$ , a parabolic ejection solution can be defined as follows  $\mathbf{r}_1(t) = \gamma t^{2/3} \vec{c}$ , where  $\gamma = (\frac{9\alpha}{2\mu})^{1/3} > 0$  is a constant and  $|\vec{c}| = 1$ .

Let  $\vec{s} = (-1, 0)$ . For given  $\varepsilon > 0$  and a unit vector  $\vec{c}$ , we fix the vector  $\mathbf{r}(\varepsilon) = \gamma \varepsilon^{2/3} \vec{c}$ , while  $\mathbf{r}(0) = q_1(0) - q_2(0) = (q_{1x}(0) - q_{2x}(0), 0) = |\mathbf{r}(0)| \vec{s} \equiv r_0 \vec{s}$  satisfies that  $r_0 \equiv |\mathbf{r}(0)| \geq 0$ . The Lagrangian action of the two-body problem is

$$(22) \quad I(\mathbf{r}(t), \dot{\mathbf{r}}(t)) = \int_0^\varepsilon \frac{\mu}{2} |\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|} dt.$$

We define the set  $V$  as follows

$$V = \left\{ \mathbf{r}(t) \in H^1([0, \varepsilon], \mathbf{R}^2) \mid \mathbf{r}(0) = |\mathbf{r}(0)| \vec{s} \equiv r_0 \vec{s}, \mathbf{r}(\varepsilon) = \gamma \varepsilon^{2/3} \vec{c}, r_0 \geq 0 \right\}.$$

After fixing the unit vector  $\vec{c}$  and the positive constant  $\varepsilon$ , we consider the following one-end free boundary value problem:

$$(23) \quad \inf_{\mathbf{r}(t) \in V} I(\mathbf{r}(t), \dot{\mathbf{r}}(t)).$$

Our result implies that if  $\langle \vec{c}, \vec{s} \rangle \neq -1$ , then there exists small enough  $\varepsilon > 0$ , such that the parabolic ejection solution is not the action minimizer of (23).

**Lemma 4.1.** *Let  $\vec{s} = (-1, 0)$  and  $\vec{c}$  be a given unit vector which satisfies  $\langle \vec{c}, \vec{s} \rangle \neq -1$ . Then there exists small enough  $\varepsilon > 0$ , such that the parabolic ejection solution  $\mathbf{r}_1(t) = \gamma t^{2/3} \vec{c}$  of the Kepler problem does not minimize the action of (22) in the Sobolev space*

$$V = \left\{ \mathbf{r} \in H^1([0, \varepsilon], \mathbf{R}^2) \mid \mathbf{r}(0) = r_0 \vec{s}, \mathbf{r}(\varepsilon) = \gamma \varepsilon^{2/3} \vec{c}, r_0 \geq 0 \right\}.$$

*Proof.* Suppose  $\mathbf{r}_1(t) = \gamma t^{\frac{2}{3}} \vec{c}$  is a minimizer of the one-end free-boundary value problem (23). We will define a path  $\mathbf{r}(t)$  satisfying  $\mathbf{r}(0) = (\frac{\varepsilon}{N_1})^{\frac{2}{3}} \vec{s}$ ,  $\mathbf{r}(T) = \gamma \varepsilon^{\frac{2}{3}} \vec{c}$  and show it has a lower action than  $\mathbf{r}_1(t)$  for big enough  $N$  and small enough  $\varepsilon$ . Set  $\alpha = m_1 m_2$ ,  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ . The action  $I_1$  of  $\mathbf{r}_1(t)$  is computed:

$$(24) \quad I_1 = \int_0^\varepsilon \frac{\mu}{2} |\dot{\mathbf{r}}_1|^2 + \frac{\alpha}{|\mathbf{r}_1|} dt = \left( \frac{2\mu\gamma^2}{3} + \frac{3\alpha}{\gamma} \right) \varepsilon^{\frac{1}{3}}.$$

For the parabolic ejection path,  $|\mathbf{r}_1(0)| = 0$ . To find a path  $\mathbf{r}(t) \in V$  with lower action, we perturb  $r_0$  so that  $r_0 > 0$  and  $I(\mathbf{r}(t), \dot{\mathbf{r}}(t)) < I_1$ .

Note that for given  $r_0 > 0$ , the Lambert theorem implies that there exists a Keplerian arc  $\mathbf{r}(t)$  such that  $\mathbf{r}(0) = r_0 \vec{s}$  and  $\mathbf{r}(\varepsilon) = \gamma \varepsilon^{\frac{2}{3}} \vec{c}$ . This Keplerian arc could be elliptic, hyperbolic or parabolic. Let  $r_0 = (\frac{\varepsilon}{N_1})^{\frac{2}{3}}$ . We then estimate the action  $I(\mathbf{r}(t), \dot{\mathbf{r}}(t))$  case by case. A detail estimate of action  $I$  implies in the three cases, the acitons are all bounded by  $I < 2\sqrt{2}(\alpha\mu\gamma)^{\frac{1}{2}} \varepsilon^{\frac{1}{3}}$ . And hence by  $\gamma = (\frac{9\alpha}{2\mu})^{1/3}$ ,

$$\Delta I := I - I_0 < 2\sqrt{2}(\alpha\mu\gamma)^{\frac{1}{2}} \varepsilon^{\frac{1}{3}} - \left( \frac{2\mu\gamma^2}{3} + \frac{3\alpha}{\gamma} \right) \varepsilon^{\frac{1}{3}} = 0.$$

**Case 1** (ellipse) We introduce polar coordinate  $(r, \theta)$ , such that  $\mathbf{r} = r e^{i\theta}$ , where  $r \geq 0$ . According to Kepler's laws, the solution satisfies:

$$r = \frac{p}{1 + e \cos \theta}, \quad \text{where } \theta \text{ is the true anomaly;}$$

or

$$r = a(1 - e \cos E), \quad \text{where } E \text{ is the eccentric anomaly.}$$

It follows that

$$\frac{dE}{d\theta} = \frac{1 - e \cos E}{\sqrt{1 - e^2}}.$$

Also, the following identity holds

$$(25) \quad \sqrt{\frac{\alpha}{\mu a^3}} (t - \tau) = E - e \sin E,$$

where  $\tau$  is a constant. Differentiate equation (25) implies

$$\sqrt{\frac{\alpha}{\mu a^3}} dt = (1 - e \cos E) dE.$$

Then the action  $I$  can be calculated as follows

$$(26) \quad \begin{aligned} I &= \int_0^\varepsilon \frac{\mu}{2} |\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|} dt = \int_0^\varepsilon \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{\alpha}{r} dt \\ &= \int_0^\varepsilon \frac{\mu}{2} [a^2 e^2 \sin^2 E + a^2 (1 - e^2)] \dot{E}^2 + \frac{\alpha}{a(1 - e \cos E)} dt \\ &= \int_{E_0}^{E_\varepsilon} \left( \frac{1}{2} + \frac{e}{2} \cos E + 1 \right) (\alpha \mu a)^{\frac{1}{2}} dE \\ &= \left[ \frac{3}{2} (E_\varepsilon - E_0) + \frac{e}{2} (\sin E_\varepsilon - \sin E_0) \right] (\alpha \mu a)^{\frac{1}{2}}, \end{aligned}$$

where  $E_0$  and  $E_\varepsilon$  are the values of  $E$  at time  $t = 0$  and  $t = \varepsilon$  respectively. Set  $\psi = \frac{E_\varepsilon - E_0}{2}$ ,  $\cos \phi = e \cos \frac{E_\varepsilon + E_0}{2}$ , we define

$$\xi = \phi + \psi, \quad \eta = \phi - \psi.$$

It follows that

$$E_\varepsilon - E_0 = \xi - \eta, \quad \sin E_\varepsilon - \sin E_0 = \frac{1}{e}(\sin \xi - \sin \eta).$$

Therefore the action  $I$  in (26) becomes

$$(27) \quad I = \left[ \frac{3}{2}(\xi - \eta) + \frac{1}{2}(\sin \xi - \sin \eta) \right] (\alpha \mu a)^{\frac{1}{2}}.$$

Note that  $|\mathbf{r}(0)| = (\frac{\varepsilon}{N_1})^{\frac{2}{3}}$  and  $|\mathbf{r}(\varepsilon)| = \gamma \varepsilon^{\frac{2}{3}}$ , and

$$|\mathbf{r}(0) - \mathbf{r}(\varepsilon)|^2 = (\frac{\varepsilon}{N_1})^{\frac{4}{3}} + \gamma^2 \varepsilon^{\frac{4}{3}} - 2(\frac{\varepsilon}{N_1})^{\frac{2}{3}} \gamma \varepsilon^{\frac{2}{3}} \langle \vec{c}, \vec{s} \rangle.$$

By Lambert's theorem,

$$(28) \quad \sqrt{\frac{\alpha}{\mu a^3}} \varepsilon = 2 [\xi - \sin \xi - (\eta - \sin \eta)],$$

where

$$(29) \quad \begin{aligned} \cos \xi &= 1 - \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| + |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{2a}, \\ \cos \eta &= 1 - \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| - |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{2a}. \end{aligned}$$

Or

$$\begin{aligned} \sin \frac{\xi}{2} &= \frac{1}{2} \left( \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| + |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right)^{\frac{1}{2}}, \\ \sin \frac{\eta}{2} &= \frac{1}{2} \left( \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| - |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right)^{\frac{1}{2}}, \end{aligned}$$

then for small enough  $\varepsilon > 0$  and big enough  $N_1 > 0$ ,

$$\begin{aligned} \frac{\xi}{2} &= \arcsin \frac{1}{2} \left( \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| + |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left[ \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| + |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right]^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{3}}), \end{aligned}$$

and

$$\begin{aligned} \frac{\eta}{2} &= \arcsin \frac{1}{2} \left( \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| - |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right)^{\frac{1}{2}}, \\ &= \frac{1}{2} \left[ \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| - |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{a} \right]^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{3}}), \end{aligned}$$



where  $|\mathbf{r}(0)| = (\frac{\varepsilon}{N})^{\frac{2}{3}}$  and  $|\mathbf{r}(\varepsilon)| = \gamma\varepsilon^{\frac{2}{3}}$ . For a fixed a small  $\varepsilon > 0$ , it is known that when  $N_1 \rightarrow \infty$ ,  $a \rightarrow \infty$ . It implies that both  $\xi$  and  $\eta$  approach 0 when  $N_1 \rightarrow \infty$ . By taking  $\varepsilon > 0$  small enough and  $N > 0$  big enough, we have

$$\begin{aligned}
 (30) \quad I &= \left[ \frac{3}{2}(\xi - \eta) + \frac{1}{2}(\sin \xi - \sin \eta) \right] (\alpha\mu a)^{\frac{1}{2}} \\
 &< 2(\xi - \eta)(\alpha\mu a)^{\frac{1}{2}} \\
 &= 2\sqrt{2}(\alpha\mu)^{\frac{1}{2}} \left[ N_1^{-\frac{2}{3}} + \gamma - \sqrt{2\gamma(1 + \langle \vec{c}, \vec{s} \rangle)} N_1^{-\frac{1}{3}} \right]^{\frac{1}{2}} \varepsilon^{\frac{1}{3}} + o(\varepsilon^{\frac{1}{3}}).
 \end{aligned}$$

Since  $\langle \vec{c}, \vec{s} \rangle \neq -1$ , it follows that for big enough  $N_1 > 0$ ,

$$(31) \quad N_1^{-\frac{2}{3}} + \gamma - \sqrt{2\gamma(1 + \langle \vec{s}, \vec{c} \rangle)} N_1^{-\frac{1}{3}} < \gamma.$$

Therefore, (30) and (31) imply that there exist small enough  $\varepsilon > 0$  and big enough  $N_1 > 0$ , such that

$$I < 2\sqrt{2}(\alpha\mu\gamma)^{\frac{1}{2}} \varepsilon^{\frac{1}{3}}.$$

**Case 2** (parabola) Let  $\mathbf{r} = re^{if}$ . The parabolic solution of the Kepler problem is

$$r = \frac{p}{1 + \cos(f)} = \frac{p}{2} \sec^2 \frac{f}{2}.$$

By setting  $\sigma = \sqrt{p} \tan \frac{f-f_0}{2}$ , the above equation becomes

$$r = \frac{p + \sigma^2}{2}.$$

We also have the Barker's equation

$$6\sqrt{\frac{\alpha}{\mu}}(t - \tau) = 3p\sigma + \sigma^3$$

with  $\tau$  a constant. Then the action of the parabolic orbit is

$$\begin{aligned}
 (32) \quad I &= \int_0^\varepsilon \frac{\mu}{2} |\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|} dt = \int_0^T \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{f}^2) + \frac{\alpha}{r} dt \\
 &= \int_0^\varepsilon \frac{\mu}{2} \left[ (\sigma\dot{\sigma})^2 + \left( \frac{p + \sigma^2}{2} \right)^2 \left( \frac{df}{d\sigma} \sigma \right)^2 \right] + \frac{2\alpha}{p + \sigma^2} dt \\
 &= \int_{\sigma_0}^{\sigma_\varepsilon} \left( \frac{\mu(\sigma^2 + p)}{2} \left( \frac{2}{p + \sigma^2} \sqrt{\frac{\alpha}{\mu}} \right)^2 + \frac{2\alpha}{p + \sigma^2} \right) \frac{p + \sigma^2}{2} \sqrt{\frac{\mu}{\alpha}} d\sigma \\
 &= 2\sqrt{\alpha\mu}(\sigma_\varepsilon - \sigma_0),
 \end{aligned}$$

where  $\sigma_0$  and  $\sigma_\varepsilon$  are the values of  $\sigma$  at  $t = 0$  and  $t = \varepsilon$  respectively. Denote  $c_1 = |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|$ ,  $s_1 = \frac{|\mathbf{r}_0| + |\mathbf{r}_T| + c}{2}$ . Then

$$p + \sigma_0\sigma_\varepsilon = 2\sqrt{s_1(s_1 - c_1)},$$

$$2(|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)|) = (\sigma_\varepsilon - \sigma_0)^2 + 2(p + \sigma_0\sigma_\varepsilon).$$

It follows that

$$\sigma_\varepsilon - \sigma_0 = \sqrt{2}(\sqrt{s_1} - \sqrt{s_1 - c_1}).$$

Therefore, by taking  $\varepsilon > 0$  small enough and  $N_1 > 0$  big enough, the action  $I$  in (32) becomes

$$\begin{aligned} I &= 2\sqrt{2\alpha\mu}(\sqrt{s_1} - \sqrt{s_1 - c_1}) \\ &= 2(2\alpha\mu)^{\frac{1}{2}} \left[ N_1^{-\frac{2}{3}} + \gamma - (2N_1^{-\frac{2}{3}}\gamma(1 + \langle \vec{c}, \vec{s} \rangle))^{\frac{1}{2}} \right]^{\frac{1}{2}} \varepsilon^{\frac{1}{3}} \\ &< 2\sqrt{2}(\alpha\mu\gamma)^{\frac{1}{2}} \varepsilon^{\frac{1}{3}}. \end{aligned}$$

**Case 3** (hyperbola) The solution of a hyperbolic orbit in true anomaly  $f$  is

$$r = \frac{p}{1 + e \cos(f - f_1)}, \quad p = a(1 - e^2),$$

where  $a < 0$  and  $e > 1$ . By choosing a suitable parameter  $H$ , it becomes

$$r = a(1 - e \cosh H).$$

From the above two equations we have

$$\frac{df}{dH} = \frac{\sqrt{e^2 - 1}}{e \cosh H - 1}.$$

And in the hyperbolic case, the following identity holds:

$$\sqrt{\frac{\alpha}{\mu(-a)^3}}(t - \tau) = e \sinh H - H.$$

The the action  $I$  satisfies

$$\begin{aligned} I &= \int_0^\varepsilon \frac{\mu}{2} |\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|} dt \\ &= \int_0^\varepsilon \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{f}^2) + \frac{\alpha}{r} dt \\ &= \int_0^\varepsilon \frac{\mu}{2} [a^2 e^2 \sinh^2 H \dot{H}^2 + a^2 (e^2 - 1) \dot{H}^2] + \frac{\alpha}{a(1 - e \cosh H)} dt \\ &= \int_{H_0}^{H_\varepsilon} \left[ \frac{1}{2} + \frac{e}{2} \cosh H + 1 \right] (-\alpha\mu a)^{\frac{1}{2}} dH \\ &= \left[ \frac{3}{2} (H_\varepsilon - H_0) + \frac{e}{2} (\sinh H_\varepsilon - \sinh H_0) \right] (-\alpha\mu a)^{\frac{1}{2}}, \end{aligned}$$

where  $H_0$  and  $H_\varepsilon$  are the values of  $H$  at  $t = 0$  and  $t = \varepsilon$  respectively. By introducing parameters  $\psi = \frac{H_\varepsilon - H_0}{2}$ ,  $\cosh \phi = e \cosh \frac{H_\varepsilon + H_0}{2}$ , it follow that  $\xi = \phi + \psi$ ,  $\eta = \phi - \psi$ . Hence

$$(33) \quad I = \left[ \frac{3}{2} (\xi - \eta) + \frac{1}{2} (\sinh \xi - \sinh \eta) \right] (-\alpha\mu a)^{\frac{1}{2}}.$$

By Lambert's theorem, we have

$$\sqrt{\frac{\alpha}{\mu(-a)^3}}\varepsilon = 2[\xi - \sinh \xi - (\eta - \sinh \eta)],$$

where

$$\cosh \xi = 1 + \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| + |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{-2a},$$

$$\cosh \eta = 1 + \frac{|\mathbf{r}(0)| + |\mathbf{r}(\varepsilon)| - |\mathbf{r}(0) - \mathbf{r}(\varepsilon)|}{-2a}.$$

Therefore, by taking  $\varepsilon > 0$  small enough and big enough  $N > 0$ , a similiar argument shows that

$$I < 2\sqrt{2}(\alpha\mu\gamma)^{\frac{1}{2}}\varepsilon^{\frac{1}{3}}.$$

The proof is complete.  $\square$

*Remark 4.2.* Assume  $\varepsilon > 0$  is small enough. If  $\langle \vec{c}, \vec{s} \rangle = -1$ , then the perturbed path is also a collision path. By the minimizing property of the collision-ejection homethetic path, it implies that the minimizer of the one-end free-boundary problem  $\inf_{\mathbf{r}(t) \in V} I(\mathbf{r}(t), \dot{\mathbf{r}}(t))$  in (23) is exactly the parabolic ejection orbit  $\mathbf{r}_1(t) = \gamma t^{\frac{2}{3}} \vec{c}$ .

As a direct application, Lemma 4.1 can be applied to show that under certain conditions, the action minimizer  $\mathcal{P}_0$  in Theorem 1.1 has no binary collision. Recall that

$$Qstart = \begin{bmatrix} q_1(a_1, \dots, a_k) \\ \dots \\ q_N(a_1, \dots, a_k) \end{bmatrix}, \quad Qend = \begin{bmatrix} q_1(b_1, \dots, b_s) \\ \dots \\ q_N(b_1, \dots, b_s) \end{bmatrix},$$

where  $Qstart, Qend \in \chi$ ,  $q_i \in \mathbf{R}^d$ , ( $i = 1, \dots, N$ ) and  $a_1, \dots, a_k, b_1, \dots, b_s$  are independent variables. Let  $(a_1, \dots, a_k, b_1, \dots, b_s) \in \mathcal{S}$ , where  $\mathcal{S}$  is a closed convex subset in  $\mathbf{R}^{k+s}$ . Assume  $q_i(0) = q_i(a_1, \dots, a_k)$  to be linear with respect to  $a_1, \dots, a_k$ , and  $q_i(1) = q_i(b_1, \dots, b_s)$  to be linear with respect to  $b_1, \dots, b_s$  ( $i = 1, \dots, N$ ).

**Theorem 4.3.** *Let  $\mathcal{S} = \mathbf{R}^{k+s}$ . If the intersection of the two configuration subsets is at origin, i.e.*

$$\{Qstart | (a_1, \dots, a_k) \in \mathbf{R}^k\} \cap \{Qend | (b_1, \dots, b_s) \in \mathbf{R}^s\} = \{\vec{0}\},$$

*the action minimizer  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  has no binary collision.*

*Proof.* The existence of action minimizer  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  is shown by Theorem 1.1. By using the result of Marchal [12] and Chenciner [2] regarding to minimizing problems with fixed ends, it follows that  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  has no collision in  $(0, 1)$ . Then we only need to show that  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  has no binary collision on the two ends. Basically, it has two types of binary collisions: a single binary collision, a simultaneous binary collision. Here we only show the case for single binary collisions by contradiction! Due to [16, 17, 24], simultaneous binary collision can be treated as several separated binary collisions. Then we can exclude the simultaneous binary collision in  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  similarly.

The following argument is standard and it can be found in [2, 4, 6, 7, 12, 21] etc.

Without loss of generality, we assume in  $\mathcal{P}_0 \in H^1([0, 1], \chi)$ , bodies 1 and 2 collide and other bodies are away from collision at  $t = 0$ . Let  $S_1$  be the minimizing path  $\mathcal{P}_0 : q^*(t) = (q_1^*(t)^T, q_2^*(t)^T, \dots, q_n^*(t)^T)^T$ . We will build the two following paths  $S_2$  (Kepler ejection orbits at the starting point) and  $S_3$  (the deformation of  $S_2$ ) with (A) Exactly the same motion of all bodies in the interval  $[\varepsilon, 1]$ . (B) At the time interval  $[0, \varepsilon]$ , the ejection orbits are replaced by a collision free orbits with boundary conditions satisfying boundary conditions of  $Qstart$  with parameters restricted in  $\mathbf{R}^k$ . Let  $\Omega_0 = \{Qstart | (a_1, \dots, a_k) \in \mathbf{R}^k\}$  be the  $k$ -dimensional linear space of  $Qstart$ . The corresponding actions will be denoted by  $A_1 = \mathcal{A}(q^*)$ ,  $A_2 = \mathcal{A}(S_2)$ ,  $A_3 = \mathcal{A}(S_3)$ . We want to prove that  $A_1 > A_3$  for sufficiently small time  $\varepsilon$ . Since (A), the actions are different only in the time interval  $[0, \varepsilon]$ .

First, we consider the ejection orbits in the starting time interval  $[0, \varepsilon]$  in  $S_2$ . Let  $r$  be the simple radial two-body motion leading from 0 to  $r_\varepsilon$  in the time interval  $[0, \varepsilon]$ . By Sundman and Sperling's estimates near collisions [19, 20], there exists a positive constant  $\gamma$  such that  $\mathbf{r}(t) = (\gamma t^{\frac{2}{3}})\vec{c}$  where  $\vec{c}$  is a unit vector in the plane. Let  $\xi(t) = \frac{m_1 q_1^*(t) + m_2 q_2^*(t)}{m_1 + m_2}$  be the center of mass of the 1st and 2nd bodies.

$$q_{1S_2}(t) = \xi(t) + \frac{m_2}{m_1 + m_2} \mathbf{r}(t); \quad q_{2S_2}(t) = \xi(t) - \frac{m_1}{m_1 + m_2} \mathbf{r}(t); \quad q_{iS_2}(t) = q_i(t), \quad \forall 3 \leq i \leq N.$$

Then the path  $S_3$  will be defined with the following form

$$q_{1S_3}(t) = \xi(t) + \frac{m_2}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{2S_3}(t) = \xi(t) - \frac{m_1}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{iS_3}(t) = q_i(t), \quad \forall 3 \leq i \leq N,$$

where  $\mathbf{r}_1(t)$  ( $t \in [0, \varepsilon]$ ) is a local deformation of  $\mathbf{r}(t)$  ( $t \in [0, \varepsilon]$ ), which satisfies

$$\begin{bmatrix} q_{1S_3}(0) \\ q_{2S_3}(0) \\ q_{3S_3}(0) \\ \vdots \\ q_{NS_3}(0) \end{bmatrix} = \begin{bmatrix} \xi(t) + \frac{m_2}{m_1 + m_2} \mathbf{r}_1(t) \\ \xi(t) - \frac{m_1}{m_1 + m_2} \mathbf{r}_1(t) \\ q_{3S_1}(0) \\ \vdots \\ q_{NS_1}(0) \end{bmatrix} \in \{Qstart | (a_1, \dots, a_k) \in \mathbf{R}^k\} \equiv \Omega_0,$$

and

$$|\mathbf{r}_1(0)| > 0, \quad \mathbf{r}_1(\varepsilon) = \mathbf{r}(\varepsilon).$$

Define  $\vec{s} = \frac{q_{1S_3}(0) - q_{2S_3}(0)}{|q_{1S_3}(0) - q_{2S_3}(0)|} = \frac{\mathbf{r}_1(0)}{|\mathbf{r}_1(0)|}$ . Note that  $\Omega_0$  is a  $k$ -dimensional linear space, and both

$$\begin{bmatrix} q_{1S_3}(0) \\ q_{2S_3}(0) \\ q_{3S_3}(0) \\ \vdots \\ q_{NS_3}(0) \end{bmatrix} = \begin{bmatrix} \xi(0) + \frac{m_2}{m_1 + m_2} \mathbf{r}_1(0) \\ \xi(0) - \frac{m_1}{m_1 + m_2} \mathbf{r}_1(0) \\ q_{3S_1}(0) \\ \vdots \\ q_{NS_1}(0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q_{1S_1}(0) \\ q_{2S_1}(0) \\ q_{3S_1}(0) \\ \vdots \\ q_{NS_1}(0) \end{bmatrix} = \begin{bmatrix} \xi(0) \\ \xi(0) \\ q_{3S_1}(0) \\ \vdots \\ q_{NS_1}(0) \end{bmatrix}$$

are in  $\Omega_0$ . It follows that

$$\begin{bmatrix} \xi(0) - \frac{m_2}{m_1 + m_2} \mathbf{r}_1(0) \\ \xi(0) + \frac{m_1}{m_1 + m_2} \mathbf{r}_1(0) \\ q_{3S_1}(0) \\ \vdots \\ q_{NS_1}(0) \end{bmatrix} \in \Omega_0.$$

Then

$$q_{1S_3}(t) = \xi(t) - \frac{m_2}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{2S_3}(t) = \xi(t) + \frac{m_1}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{iS_3}(t) = q_i(t), \quad \forall 3 \leq i \leq N$$

also satisfies that  $\begin{bmatrix} q_{1S_3}(0) \\ q_{2S_3}(0) \\ q_{3S_3}(0) \\ \vdots \\ q_{NS_3}(0) \end{bmatrix} \in \Omega_0$ . In other words, in the definition of  $S_3$ , both  $\mathbf{r}_1(t)$  and  $-\mathbf{r}_1(t)$  satisfy

the boundary condition  $\Omega_0$ . The rest of the proof is to show there exists some  $\mathbf{r}_1(t)$  such that  $A_3 < A_1$ . The actions are computed by two parts: the first part  $A_{in}$  is the action of the collision subsystem involving colliding bodies  $m_1$  and  $m_2$ ; the second part  $A_{out}$  is the action of the remainder. By direct computation, we have

$$A_{2in} - A_{3in} = \int_0^\varepsilon \frac{m_1 m_2}{2(m_1 + m_2)} (|\dot{\mathbf{r}}|^2 - |\dot{\mathbf{r}}_1|^2) + m_1 m_2 \left( \frac{1}{|\mathbf{r}|} - \frac{1}{|\mathbf{r}_1|} \right) dt,$$

$$A_{3out} - A_{1out} = \int_0^\varepsilon \sum_{i=1,2} \left( \frac{m_i m_3}{|q_i - q_{3S_3}|} - \frac{m_i m_3}{|q_i - q_3|} \right) dt.$$

It is known that  $A_{1in} \geq A_{2in}$  since the homothetic parabolic collision-ejection orbit is a minimizer. In order to prove  $A_1 > A_3$  in  $[0, \varepsilon]$  we shall prove that  $A_{2in} - A_{3in} > A_{3out} - A_{1out}$ .

We first estimate the bounds for  $A_{out}$ . Consider the motion of the  $j$ -th body between the arbitrary successive instants  $t_1$  and  $t_2$ . Because the minimum of the integral  $\int_{t_1}^{t_2} \frac{m_j |\dot{q}_j|^2}{2} dt$  between given positions  $q_j(t_1)$  and  $q_j(t_2)$  is obtained for a constant velocity vector, we can always write  $\frac{m_j |q_j(t_2) - q_j(t_1)|^2}{2(t_2 - t_1)} \leq \int_{t_1}^{t_2} \frac{m_j |\dot{q}_j|^2}{2} dt \leq \mathcal{A}(q) \leq K < \infty$ . So if  $0 \leq t_1 \leq t_2 \leq 1$ ,  $|q_j(t_2) - q_j(t_1)| \leq \left( \frac{2K(t_2 - t_1)}{m_j} \right)^{1/2}$ . Pick up  $\varepsilon > 0$  small such that the two bodies  $m_1$  and  $m_2$  will remain closely around the collision point  $q_1(0)$  in the time interval  $[0, \varepsilon]$ , i.e.  $|q_1 - q_2| \leq J\sqrt{\varepsilon}$ , where  $J = 2(2K)^{1/2}$ .  $m_3$  will remain outside of the circle centered at the collision point with radius  $D$  and  $J\sqrt{\varepsilon} \leq J\sqrt{\varepsilon_0} \ll D$  for a fixed  $\varepsilon_0$ . So during the time interval  $[0, \varepsilon]$ , the body  $m_3$  is outside of the circle with radius  $D$  and center  $q_1(0)$ , while the bodies  $m_1$  and  $m_2$  are inside the much smaller circle of the same center and radius  $J\sqrt{\varepsilon}$ .

$$\begin{aligned} |A_{3out} - A_{1out}| &\leq \int_0^\varepsilon \sum_{i=1,2} \sum_{j=3}^N m_i m_j \left| \left( \frac{|q_i - q_{3S_3}| - |q_i - q_j|}{|q_i - q_j| |q_i - q_{jS_3}|} \right) \right| dt \\ &\leq \int_0^\varepsilon \sum_{i=1,2} \sum_{j=3}^N m_i m_j \left( \frac{|q_{jS_3} - q_3|}{|q_i - q_j| |q_i - q_{jS_3}|} \right) dt \\ (34) \quad &\leq \int_0^\varepsilon \sum_{i=1,2} \sum_{j=3}^N m_i m_j \left( \frac{J\sqrt{\varepsilon}}{(D - J\sqrt{\varepsilon_0})^2} \right) dt = \frac{2J}{(D - J\sqrt{\varepsilon_0})^2} \varepsilon^{\frac{3}{2}} = U_{out} \varepsilon^{\frac{3}{2}}. \end{aligned}$$

Then we estimate  $A_{in}$ . By Lemma 4.1, if  $\langle \vec{c}, \vec{s} \rangle \neq -1$ , for small enough  $\varepsilon > 0$ , there exists a local deformation  $\mathbf{r}_1$ , such that at  $t = 0$ ,  $|\mathbf{r}_1(0)| > 0$  and

$$A_{2in} > A_{3in}, \quad \text{and} \quad A_{2in} - A_{3in} = O(\varepsilon^{\frac{1}{3}}).$$

If  $\langle \vec{c}, \vec{s} \rangle = -1$ , we can choose the deformation to be  $-\mathbf{r}_1(t)$  and  $S_3$  is defined as

$$q_{1S_3}(t) = \xi(t) - \frac{m_2}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{2S_3}(t) = \xi(t) + \frac{m_1}{m_1 + m_2} \mathbf{r}_1(t); \quad q_{iS_3}(t) = q_i(t), \quad \forall 3 \leq i \leq N.$$

In this case,  $\vec{s}$  becomes  $-\vec{s}$  and it satisfies the assumption  $\langle \vec{c}, -\vec{s} \rangle \neq -1$ . Hence, there always exists a deformation  $\mathbf{r}_1(t)$  of  $\mathbf{r}(t)$  such that

$$(35) \quad A_{2in} > A_{3in}, \quad \text{and} \quad A_{2in} - A_{3in} = O(\varepsilon^{\frac{1}{3}})$$

Therefore, by inequalities (34) and (35), there exists small enough  $\varepsilon > 0$ , such that

$$A_{2in} - A_{3in} > A_{3out} - A_{1out}.$$

It implies that  $A_1 > A_3$ . Contradiction! So under the assumption of Theorem 1.1 with  $S = \mathbb{R}^{k+s}$ , there is no binary collision in the action minimizer  $\mathcal{P}_0$ . The proof is complete.  $\square$

Similarly, if  $Qend = \begin{bmatrix} q_1(b_1, \dots, b_s) \\ \dots \\ q_N(b_1, \dots, b_s) \end{bmatrix} R(\theta)$ , the following result holds.

**Corollary 4.4.** *Let  $S = \mathbb{R}^{k+s+1}$ . If the intersection of the two configuration subsets is at origin, i.e.*

$$\{Qstart | (a_1, \dots, a_k) \in \mathbb{R}^k\} \cap \{Qend | (b_1, \dots, b_s, \theta) \in \mathbb{R}^{s+1}\} = \{\vec{0}\},$$

*the action minimizer  $\mathcal{P}_0 \in H^1([0, 1], \chi)$  has no binary collision.*

In the end of this section, we apply Theorem 4.3 to our minimizing problem:

$$(36) \quad \mathcal{A}(\mathcal{P}) = \inf_{\{(a_1, a_2, b_1, b_2) \in S_1\}} \inf_{\{q(0)=Qstart_1, q(1)=Qend_1, q(t) \in H^1([0, 1], \chi)\}} \mathcal{A},$$

where  $\mathcal{A} = \int_0^1 (K + U) dt$ ,

$$(37) \quad Qstart_1 = \begin{bmatrix} -2a_1 - a_2 & 0 \\ a_1 - a_2 & 0 \\ a_1 + 2a_2 & 0 \end{bmatrix}, \quad Qend_1 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and  $(a_1, a_2, b_1, b_2) \in S_1$ , with

$$(38) \quad S_1 = \{a_1 \geq 0, \quad a_2 \geq 0, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}\}.$$

Theorem 4.3 implies that the minimizer  $\mathcal{P}$  has no binary collision at  $t = 1$ . However, at  $t = 0$ , the binary collision can NOT be excluded by local deformation. Assume body  $i$  has mass  $m_i$  with coordinate  $q_i (i = 1, 2, 3)$ . For a binary collision of bodies  $m_i$  and  $m_j$ , we define the direction of collision  $\vec{c}_{ij}$  to be

$$\vec{c}_{ij} = \lim_{t \rightarrow 0^+} \frac{\mathbf{r}_{ij}(t)}{|\mathbf{r}_{ij}(t)|}$$

where  $\mathbf{r}_{ij} = q_i - q_j$ . By applying Lemma 4.1, a similar argument as in Theorem 4.3 shows that

**Corollary 4.5.** *If the minimizer  $\mathcal{P}$  has a binary collision between bodies 1 and 2 at  $t = 0$ , then the direction of collision  $\vec{c}_{12}$  must be  $(1, 0)$ . If the minimizer  $\mathcal{P}$  has a binary collision between bodies 2 and 3 at  $t = 0$ , then the direction of collision  $\vec{c}_{23}$  must be  $(1, 0)$ .*

*Proof.* Since  $a_1 \geq 0$  and  $a_2 \geq 0$ , it implies that the three bodies lie on the  $x$ -axis at  $t = 0$  with an order  $q_{1x}(0) \leq q_{2x}(0) \leq q_{3x}(0)$ . If bodies 1 and 2 collide at  $t = 0$ , then  $\vec{s} = (-1, 0)$ . If  $\vec{c}_{12} \neq (1, 0)$ , that is,  $\langle \vec{c}, \vec{s} \rangle \neq -1$ , Lemma 4.1 and the proof of Theorem 4.3 imply that there exist a local deformation which can lower the action of  $\mathcal{P}$ . Contradiction! Therefore, if there is a binary collision between bodies 1 and 2 in the minimizer  $\mathcal{P}$ ,  $\vec{c}_{12}$  must be  $(1, 0)$ . Similarly, if bodies 2 and 3 collide at  $t = 0$  in  $\mathcal{P}$ ,  $\vec{c}_{23}$  must be  $(1, 0)$ . The proof is complete.  $\square$

## 5 Analysis of binary collisions at $t = 0$

In this section, we prove that in the minimizer  $\mathcal{P}$ , the only possible collision is the binary collision between bodies 1 and 2 at  $t = 0$ . And the corresponding solution must be the Schubart orbit in Fig. 2.

### 5.1 Exclusion of binary collision between bodies 2 and 3 and $t = 0$

We show that if bodies 2 and 3 collide at  $t = 0$  in the minimizer  $\mathcal{P}$ , the action  $\mathcal{A}(\mathcal{P}) \geq 4.21617$ . By Lemma 3.1, there exists a testing path  $\hat{q}(t)$  which has action  $\mathcal{A}(\hat{q}(t)) < 3.5383$ . Therefore,  $\mathcal{P}$  can not have collision between bodies 2 and 3 at  $t = 0$ .

**Lemma 5.1.** *The minimizer  $\mathcal{P}$  has no binary collision between bodies 2 and 3 at  $t = 0$ .*

*Proof.* Assume that bodies 2 and 3 collide at  $t = 0$ . Note that  $\mathcal{P}$  is the action minimizer of (36), with  $Qstart_1$ ,  $Qend_1$  and  $S_1$  defined by (37) and (38). It follows that  $\mathcal{P}$  is also an action minimizer under the following settings:

$$(39) \quad \inf_{\{(a_1, b_1, b_2) \in S_2\}} \inf_{\{q(0)=Qstart_2, q(1)=Qend_2, q(t) \in H^1([0,1], \chi)\}} \mathcal{A},$$

where  $\mathcal{A} = \int_0^1 (K + U) dt$ ,

$$(40) \quad Qstart_2 = \begin{bmatrix} -2a_1 & 0 \\ a_1 & 0 \\ a_1 & 0 \end{bmatrix}, \quad Qend_2 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and  $(a_1, b_1, b_2) \in S_2$ , with

$$(41) \quad S_2 = \{a_1 \geq 0, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}\}.$$

By assumption, the action  $\mathcal{A}(\mathcal{P})$  is the minimum action for the minimizing problem (39) with  $Qstart_2$ ,  $Qend_2$  and  $S_2$  defined in (40) and (41). To find a lower bound of  $\mathcal{A}(\mathcal{P})$ , we define a new functional space such that  $\mathcal{P}$  is in this space. Let

$$(42) \quad Qstart_3 = \begin{bmatrix} -2a_1 & 0 \\ a_1 & c_1 \\ a_1 & -c_1 \end{bmatrix}, \quad Qend_3 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and  $(a_1, c_1, b_1, b_2) \in \mathbb{R}^4$ . We consider the following minimizing problem

$$(43) \quad \inf_{\{(a_1, c_1, b_1, b_2) \in \mathbb{R}^4\}} \inf_{\{q(0)=Qstart_3, q(1)=Qend_3, q(t) \in H^1([0,1], \chi)\}} \mathcal{A}.$$

By Theorem 1.1, there exists a minimizer  $\tilde{\mathcal{P}}$ , such that

$$\mathcal{A}(\tilde{\mathcal{P}}) = \inf_{\{(a_1, c_1, b_1, b_2) \in \mathbb{R}^4\}} \inf_{\{q(0)=Qstart_3, q(1)=Qend_3, q(t) \in H^1([0,1], \chi)\}} \mathcal{A}.$$

Note that under the assumption, it is clear that  $\mathcal{A}(\mathcal{P}) \geq \mathcal{A}(\tilde{\mathcal{P}})$ . The rest of the proof is to find a lower bound for  $\mathcal{A}(\tilde{\mathcal{P}})$ .

By Theorem 4.3, the minimizer  $\tilde{\mathcal{P}}$  has no binary collision. Since the circular Lagrangian orbit satisfies the boundary conditions  $Qstart_3$  and  $Qend_3$ , it follows that

$$(44) \quad \mathcal{A}(\tilde{\mathcal{P}}) \leq \frac{1}{4} \mathcal{A}_{Lagrangian} = 3 \times \frac{3}{2} (2\pi)^{\frac{2}{3}} \frac{\sqrt{3}^{\frac{2}{3}}}{3} 4^{-\frac{2}{3}} \approx 4.21617,$$

where  $\mathcal{A}_{Lagrangian}$  is the action of one period of the Lagrangian circular orbit with period 4. By inequality (17) in Section 3, the actions of the paths of total collision have a lower bound  $\mathcal{A}_{total\_collision} \geq 6.6927$ . It follows that  $\tilde{\mathcal{P}}$  has no total collision. Therefore, the minimizer  $\tilde{\mathcal{P}}$  is a classical solution of the planar three-body problem.

We will show that  $\tilde{\mathcal{P}}(t \in [0, 1])$  can be extended to a periodic solution with period  $T = 4$ . Let  $\tilde{q}_i(t) = (q_{ix}(t), q_{iy}(t))$  ( $i = 1, 2, 3, t \in [0, 1]$ ) be the coordinates of each body in  $\tilde{\mathcal{P}}$ . By formulas of first variation, it is known that

$$(45) \quad \dot{\tilde{q}}_{1x}(0) = 0, \quad \dot{\tilde{q}}_{2x}(0) = -\dot{\tilde{q}}_{3x}(0), \quad \dot{\tilde{q}}_{2y}(0) = \dot{\tilde{q}}_{3y}(0);$$

$$(46) \quad \dot{\tilde{q}}_{1y}(1) = 0, \quad \dot{\tilde{q}}_{2y}(1) = -\dot{\tilde{q}}_{3y}(1), \quad \dot{\tilde{q}}_{2x}(1) = \dot{\tilde{q}}_{3x}(1).$$

By the uniqueness of solutions of the initial value problem in an ODE system, the minimizer  $\tilde{\mathcal{P}}(t \in [0, 1])$  can be extended as follows

$$(47) \quad \begin{cases} \tilde{q}_1(t) = (-\tilde{q}_{1x}(2-t), \tilde{q}_{1y}(2-t)), & t \in [1, 2]; \\ \tilde{q}_2(t) = (-\tilde{q}_{3x}(2-t), \tilde{q}_{3y}(2-t)), & t \in [1, 2]; \\ \tilde{q}_3(t) = (-\tilde{q}_{2x}(2-t), \tilde{q}_{2y}(2-t)), & t \in [1, 2]; \\ \tilde{q}_1(t) = (-\tilde{q}_{1x}(t-2), -\tilde{q}_{1y}(t-2)), & t \in [2, 4]; \\ \tilde{q}_2(t) = (-\tilde{q}_{2x}(t-2), -\tilde{q}_{2y}(t-2)), & t \in [2, 4]; \\ \tilde{q}_3(t) = (-\tilde{q}_{3x}(t-2), -\tilde{q}_{3y}(t-2)), & t \in [2, 4]. \end{cases}$$

Let  $\tilde{q}(t) = \begin{bmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{bmatrix}$  and  $q(t) = \begin{bmatrix} q_{1x}(t) & q_{1y}(t) \\ q_{2x}(t) & q_{2y}(t) \\ q_{3x}(t) & q_{3y}(t) \end{bmatrix}$ . In [11], Long and Zhang showed that in the planar three-body problem, the action minimizer in the loop space

$$\Omega = \left\{ q(t) \in H^1(\mathbb{R}, \chi) \mid q(t) = q(t+T), \quad q(t) = -q(t+T/2), \quad \forall t \in \mathbb{R} \right\}$$



must be the non-collision equilateral triangle circular periodic solutions. And its action is

$$\mathcal{A}_{Lagrangian} = 3 \times \frac{3}{2} (2\pi)^{\frac{2}{3}} \frac{\sqrt{3}^{\frac{2}{3}}}{3} 4^{\frac{1}{3}} \approx 16.8647.$$

Since  $\tilde{q}(t) \in \Omega$ , it follows that

$$(48) \quad \mathcal{A}(\tilde{q}(t) (t \in [0, 1])) = \mathcal{A}(\tilde{\mathcal{P}}) \geq \frac{1}{4} \mathcal{A}_{Lagrangian} \approx 4.21617.$$

Inequalities (44) and (48) imply that

$$\mathcal{A}(\tilde{\mathcal{P}}) = \frac{1}{4} \mathcal{A}_{Lagrangian} \approx 4.21617.$$

Therefore,

$$\mathcal{A}(\mathcal{P}) \geq \mathcal{A}(\tilde{\mathcal{P}}) \approx 4.21617.$$

By Corollary 3.2, the lower bound of the minimizer  $\mathcal{P}$  is  $\mathcal{A}(\mathcal{P}) < 3.5383 < 4.21617$ . Contradiction!

Hence, the minimizer  $\mathcal{P}$  has no collision between bodies 2 and 3 at  $t = 0$ . The proof is complete.  $\square$

## 5.2 Binary collision between bodies 1 and 2 at $t = 0$

By Theorem 4.3 and Lemma 5.1, the only possible collision in the minimizer  $\mathcal{P}$  is the collision between bodies 1 and 2 at  $t = 0$ . In this subsection, we show that in the minimizer  $\mathcal{P}$ , it is possible to have a binary collision between bodies 1 and 2 at  $t = 0$ . And we show that  $\mathcal{P}$  must coincide with the Schubart orbit [18] in Fig. 2, which is shown to exist in [13, 22].

Assume bodies 1 and 2 collide at  $t = 0$ , it follows that the minimizer  $\mathcal{P}$  is also a minimizer under the following settings:

$$(49) \quad \mathcal{A}(\mathcal{P}) = \inf_{\{(a_1, b_1, b_2) \in S_4\}} \inf_{\{q(0)=Qstart_4, q(1)=Qend_4, q(t) \in H^1([0, 1], \mathcal{X})\}} \mathcal{A},$$

$$Qstart_4 = \begin{bmatrix} -a_2 & 0 \\ -a_2 & 0 \\ 2a_2 & 0 \end{bmatrix}, \quad Qend_4 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

and

$$(50) \quad S_4 = \{a_2 \geq 0, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}\}.$$

In order to find the minimizer  $\mathcal{P}$ , we consider an action minimizing problem in a larger functional space. Let

$$Qstart_5 = \begin{bmatrix} -a_2 & 0 \\ -a_2 & 0 \\ 2a_2 & 0 \end{bmatrix}, \quad Qend_5 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ -b_2 & b_1 \end{bmatrix} R(\theta) = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ -b_2 & b_1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},$$

and

$$S_5 = \{a_2 \in \mathbb{R}, \quad b_1 \in \mathbb{R}, \quad b_2 \in \mathbb{R}, \quad \theta \in \mathbb{R}\}.$$

The following minimizing problem is studied

$$(51) \quad \inf_{\{(a_1, b_1, b_2, \theta) \in S_5\}} \inf_{\{q(0)=Q_{start_5}, q(1)=Q_{end_5}, q(t) \in H^1([0,1], \chi)\}} \mathcal{A}.$$

By Lemma 1.2, there exists an action minimizer  $\mathcal{P}_5$ , such that

$$\mathcal{A}(\mathcal{P}_5) = \inf_{\{(a_1, b_1, b_2, \theta) \in S_5\}} \inf_{\{q(0)=Q_{start_5}, q(1)=Q_{end_5}, q(t) \in H^1([0,1], \chi)\}} \mathcal{A}.$$

By Lemma 3.1, there exists a testing path  $\hat{q}(t) \in H^1([0,1], \chi)$  which satisfies the boundary conditions  $Q_{start_5}$  and  $Q_{end_5}$ , and

$$\mathcal{A}(\hat{q}(t)) < 3.5383.$$

According to inequalities (16) and (17) in Section 3, the lower bound of the action of the total collision paths is 6.6927. It follows that  $\mathcal{P}_5$  has no total collisions. By Theorem 4.3,  $\mathcal{P}_5$  has no binary collision at  $t = 1$ . According to the regularization theory of single binary collision ([13, 22, 23]),  $\mathcal{P}_5$  can be treated as a solution of the N-body problem.

If  $\mathcal{P}$  has collision between bodies 1 and 2 at  $t = 0$ , it is clear that  $\mathcal{P}$  satisfies (49) and

$$(52) \quad \mathcal{A}(\mathcal{P}) \geq \mathcal{A}(\mathcal{P}_5).$$

In what follows, we show that  $\mathcal{P}_5$  coincides with the Schubart orbit in [18].

**Lemma 5.2.** *The angular momentum of  $\mathcal{P}_5$  is 0.*

*Proof.* We apply the first variation formula on  $\theta$ . It is known that

$$\begin{aligned} 0 &= \frac{\partial \mathcal{A}}{\partial \theta} = \frac{\partial}{\partial \theta} \int_0^1 (K + U) dt \\ &= \int_0^1 \sum_{i=1}^3 m_i \left\langle \dot{q}_i, \frac{\partial \dot{q}_i}{\partial \theta} \right\rangle + \sum_{i=1}^3 \left\langle \frac{\partial U}{\partial q_i}, \frac{\partial q_i}{\partial \theta} \right\rangle dt \\ &= \sum_{i=1}^3 m_i \left\langle \dot{q}_i, \frac{\partial q_i}{\partial \theta} \right\rangle \Big|_0^1 + \int_0^1 \sum_{i=1}^3 \left\langle -m_i \ddot{q}_i + \frac{\partial U}{\partial q_i}, \frac{\partial q_i}{\partial \theta} \right\rangle dt \\ &= \sum_{i=1}^3 m_i \left\langle \dot{q}_i, \frac{\partial q_i}{\partial \theta} \right\rangle \Big|_0^1 \\ &= 2b_1 \dot{q}_{1x}(1) \cos \theta + 2b_1 \dot{q}_{1y}(1) \sin \theta + \dot{q}_{2x}(1) (b_2 \sin \theta - b_1 \cos \theta) \\ &\quad + \dot{q}_{2y}(1) (-b_2 \cos \theta - b_1 \sin \theta) + \dot{q}_{3x}(1) (-b_2 \sin \theta - b_1 \cos \theta) \\ &\quad + \dot{q}_{3y}(1) (b_2 \cos \theta - b_1 \sin \theta), \end{aligned}$$

which implies that the angular momentum at  $t = 1$  is 0. Since angular momentum is a first integral, it follows that for any time in  $[0, 1]$ , the minimizer  $\mathcal{P}_5$  has angular momentum 0.  $\square$

**Corollary 5.3.** *The minimizer  $\mathcal{P}_5$  is a 1-dimensional motion. That is, for any  $t \in [0, 1]$ , the three bodies are always on the  $x$ -axis.*

*Proof.* By Corollary 4.5,

$$\vec{c}_{12} = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{r}_{12}(\varepsilon)}{|\mathbf{r}_{12}(\varepsilon)|} = (1, 0).$$

Next, we apply Lemma 5.2 to show that the  $y$  velocity of body 3  $\dot{q}_{3y}(0)$  must be 0. If not, by the analysis of binary collision, the collision pair will have a velocity  $\frac{1}{2}\dot{q}_{3y}(0)$  on the  $y$  axis. Therefore, the angular momentum is not 0. Contradiction! Hence,  $\dot{q}_{3y}(0) = 0$ . By Lemma 4.1, the collision pair share a common  $y$ -velocity. It implies that  $\dot{q}_{1y}(0) = \dot{q}_{2y}(0) = 0$ . Note that at  $t = 0$ , bodies 1 and 2 have a binary collision, which can be regularized by a Levi-Civita transformation [22]. It follows that in a small interval  $(0, \varepsilon)$ , the three bodies satisfy

$$\dot{q}_{1y}(t) = \dot{q}_{2y}(t) = \dot{q}_{3y}(t) = 0, \quad t \in (0, \varepsilon).$$

Therefore, by the uniqueness of solution of ODE system, it follows that

$$\dot{q}_{1y}(t) = \dot{q}_{2y}(t) = \dot{q}_{3y}(t) = 0, \quad t \in (0, 1].$$

□

**Theorem 5.4.** *The minimizer  $\mathcal{P}_5$  coincides with the Schubart orbit. Hence, if the minimizer  $\mathcal{P}$  has collision singularities, it must be the Schubart orbit.*

*Proof.* By Corollary 5.3, it follows that in the minimizer  $\mathcal{P}_5$ , all the three bodies must be on the  $x$ -axis

for all  $t \in [0, 1]$ . In particular, at  $t = 1$ , the configuration  $Qend_5 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ -b_2 & b_1 \end{bmatrix} R(\theta)$  stays on the  $x$ -axis and has no collision, which implies that  $\theta = 0$  and  $b_1 = 0$ . In other words,  $Qend_5$  becomes

$$Qend_5 = \begin{bmatrix} 0 & 0 \\ b_2 & 0 \\ -b_2 & 0 \end{bmatrix}.$$

By [22], the minimizer connecting  $Qstart_5 = \begin{bmatrix} -a_2 \\ -a_2 \\ 2a_2 \end{bmatrix}$  and  $Qend_5 = \begin{bmatrix} 0 \\ b_2 \\ -b_2 \end{bmatrix}$  on the  $x$ -axis is one forth part of the Schubart orbit. Therefore, the minimizer  $\mathcal{P}_5$  coincides with the Schubart orbit. It is clear that the Schubart orbit is a solution connecting  $Qstart_4$  and  $Qend_4$ , which implies that

$$\mathcal{A}(\mathcal{P}) \leq \frac{1}{4} \mathcal{A}_{Schubart} = \mathcal{A}(\mathcal{P}_5).$$

By inequality (52),  $\mathcal{A}(\mathcal{P}) \geq \mathcal{A}(\mathcal{P}_5)$ . It follows that

$$\mathcal{A}(\mathcal{P}) = \mathcal{A}(\mathcal{P}_5) = \frac{1}{4} \mathcal{A}_{Schubart},$$

and the minimizer  $\mathcal{P}$  must be the Schubart orbit.

□

## 6 First variation and minimizing path extension

This section shows that if the minimizer  $\mathcal{P}$  in the minimizing problem (36) has no collision, it can be extended to the Broucke-Hénon orbit in Fig. 1. Recall that

$$Qstart_1 = \begin{bmatrix} -2a_1 - a_2 & 0 \\ a_1 - a_2 & 0 \\ a_1 + 2a_2 & 0 \end{bmatrix}, \quad Qend_1 = \begin{bmatrix} 0 & -2b_1 \\ -b_2 & b_1 \\ b_2 & b_1 \end{bmatrix},$$

where  $q = (q_1^T, q_2^T, q_3^T)^T$  and  $q_i = (q_{ix}, q_{iy})$  ( $i = 1, 2, 3$ ) are row vectors in  $\mathbf{R}^2$ . The set  $\mathcal{S}_1$  is

$$\mathcal{S}_1 = \left\{ (a_1, a_2, b_1, b_2) \mid a_1 \geq 0, a_2 \geq 0, b_1 \in \mathbf{R}, b_2 \in \mathbf{R} \right\}.$$

According to Theorem 1.1, in path  $\widetilde{\mathcal{P}}_1$ , the parameters  $a_{10}, a_{20}, b_{10}, b_{20} \in \mathcal{S}$  are all finite. The boundary of  $\mathcal{S}_1$ :  $a_2 = 0$  or  $3a_1 + a_2 = 0$  corresponds to collisions. The assumption that path  $\mathcal{P}$  is collision-free implies that  $a_{10}, a_{20}$  are away from the boundary of  $\mathcal{S}_1$ . Hence, the first variation formula can be applied to the minimizing path  $\mathcal{P}$ .

**Lemma 6.1.** *Let  $q^*(t)$  ( $t \in [0, 1]$ ) be the positions of the action minimizer of  $\mathcal{P}$ . Let  $q_i^*(t)$  be the position of the  $i$ -th body and  $\dot{q}_i^*(t) = (\dot{q}_{ix}^*(t), \dot{q}_{iy}^*(t))$  be its corresponding velocity. Then*

$$(53) \quad \dot{q}_{1x}^*(0) = \dot{q}_{2x}^*(0) = \dot{q}_{3x}^*(0) = 0,$$

$$(54) \quad \dot{q}_{1y}^*(1) = 0, \quad \dot{q}_{2y}^*(1) = -\dot{q}_{3y}^*(1),$$

$$(55) \quad \dot{q}_{2x}^*(1) = \dot{q}_{3x}^*(1).$$

*Proof.* The proof of (53), (54) and (55) are essentially the same. Here we only show (53) in detail.

Let  $\xi(0)$  be an admissible variation, which means that  $q^* + \delta\xi \in \mathcal{P}(Qstart, Qend)$  for small enough  $\delta$ . The first variation  $\delta_\xi \mathcal{A}(q^*)$  satisfies

$$\begin{aligned} & \delta_\xi \mathcal{A}(q^*) \\ &= \lim_{\delta \rightarrow 0} \frac{\mathcal{A}(q^* + \delta\xi) - \mathcal{A}(q^*)}{\delta} \\ &= \int_0^1 \sum_{i=1}^3 \frac{m_i |\dot{q}_i^* + \delta\dot{\xi}_i|^2 - m_i |\dot{q}_i^*|^2}{2\delta} + \frac{U(q^* + \delta\xi) - U(q^*)}{\delta} dt \\ &= \int_0^1 \sum_{i=1}^3 m_i \langle \dot{q}_i^*, \dot{\xi}_i \rangle + \sum_{i=1}^3 \left\langle \frac{\partial U}{\partial q_i^*}, \xi_i \right\rangle dt \\ &= \sum_{i=1}^3 m_i \langle \dot{q}_i^*, \xi_i \rangle \Big|_0^T + \int_0^1 \sum_{i=1}^3 \left\langle -m_i \ddot{q}_i^* + \frac{\partial U}{\partial q_i^*}, \xi_i \right\rangle dt \\ &= \sum_{i=1}^3 m_i \langle \dot{q}_i^*, \xi_i \rangle \Big|_0^T. \end{aligned}$$

Because the first variation vanishes for any  $\xi$  and the minimizing path  $q^*$ ,  $\sum_{i=1}^3 m_i \langle \dot{q}_i^*, \xi_i \rangle \Big|_0^T = 0$ .

In particular,  $\xi(0)$  can be  $\begin{pmatrix} -2 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$  and  $\xi(1)$  can be 0. It follows that

$$(56) \quad 2\dot{q}_{1x}^*(0) - \dot{q}_{2x}^*(0) - \dot{q}_{3x}^*(0) = 0.$$

We can set  $\xi(0) = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 2 & 0 \end{pmatrix}$  and  $\xi(1) = 0$ , it follows that

$$(57) \quad -\dot{q}_{1x}^*(0) - \dot{q}_{2x}^*(0) + 2\dot{q}_{3x}^*(0) = 0.$$

Note that the total linear momentum  $\dot{q}_{1x}^*(0) + \dot{q}_{2x}^*(0) + \dot{q}_{3x}^*(0) = 0$ . It follows that  $\dot{q}_{1x}^*(0) = \dot{q}_{2x}^*(0) = \dot{q}_{3x}^*(0) = 0$ .

The proofs of the other three identities (54) and (55) can be shown by similar arguments. The proof is complete.  $\square$

With the help of Lemma 6.1, we can show that  $q^*(t) (t \in [0, 1])$  can be extended to a periodic solution.

**Lemma 6.2.**  $q^*(t) (t \in [0, 1])$  can be extended to a periodic orbit with period  $T = 4$ .

*Proof.* At  $t = 0$ , three bodies line up on the  $x$ -axis in an order  $q_1(0) < q_2(0) < q_3(0)$ . At  $t = 1$ , they form an isosceles triangle with symmetry axis on the  $y$ -axis. By Lemma 6.1,  $\dot{q}_{1x}^*(0) = \dot{q}_{2x}^*(0) = \dot{q}_{3x}^*(0) = 0$

and  $\dot{q}_{1y}^*(1) = 0, \dot{q}_{2x}^*(1) = \dot{q}_{3x}^*(1)$ . Let  $q^*(t) = \begin{bmatrix} q_1^*(t) \\ q_2^*(t) \\ q_3^*(t) \end{bmatrix} = \begin{bmatrix} q_{1x}^*(t) & q_{1y}^*(t) \\ q_{2x}^*(t) & q_{2y}^*(t) \\ q_{3x}^*(t) & q_{3y}^*(t) \end{bmatrix}$ , where  $t \in [0, 1]$ . When  $t \in (1, 2]$ , we define

$$(58) \quad q^*(t) = \begin{bmatrix} q_1^*(t) \\ q_2^*(t) \\ q_3^*(t) \end{bmatrix} = \begin{bmatrix} -q_{1x}^*(2-t) & q_{1y}^*(2-t) \\ -q_{3x}^*(2-t) & q_{3y}^*(2-t) \\ -q_{2x}^*(2-t) & q_{2y}^*(2-t) \end{bmatrix}, \quad t \in (1, 2].$$

When  $t$  approaches 1, it is easy to check that  $\lim_{t \rightarrow 1^-} q^*(t) = \lim_{t \rightarrow 1^+} q^*(t) = q^*(1)$ . On the other hand, by applying Lemma 6.1, it follows that

$$\dot{q}^*(1) = \begin{bmatrix} \dot{q}_{1x}^*(1) & -\dot{q}_{1y}^*(1) \\ \dot{q}_{3x}^*(1) & -\dot{q}_{3y}^*(1) \\ \dot{q}_{2x}^*(1) & -\dot{q}_{2y}^*(1) \end{bmatrix} = \begin{bmatrix} \dot{q}_{1x}^*(1) & 0 \\ \dot{q}_{2x}^*(1) & \dot{q}_{2y}^*(1) \\ \dot{q}_{3x}^*(1) & \dot{q}_{3y}^*(1) \end{bmatrix}.$$

Hence, at  $t = 1$ ,  $q^*(t) (t \in (1, 2])$  and  $q^*(t) (t \in [0, 1])$  are smoothly connected. By the uniqueness of solution of ODE system,  $q^*(t)$  can be extended to  $[0, 2]$  by (58). At  $t = 2$ ,

$$q_1^*(2) = (a_{10} + a_{20}, 0), \quad q_2^*(2) = (-2a_{10} - a_{20}, 0), \quad q_3^*(2) = (a_{10}, 0).$$

Note that  $a_{20} > 0$  and  $3a_{10} + a_{20} > 0$ . It follows that, at  $t = 2$  the three bodies line up on the  $x$ -axis again in an order  $q_2(2) < q_3(2) < q_1(2)$ . By Lemma 6.1, the velocities of the three bodies at  $t = 2$  are all vertical. Therefore, we can extend the path to  $t \in (2, 4]$  as follows

$$(59) \quad q^*(t) = \begin{bmatrix} q_1^*(t) \\ q_2^*(t) \\ q_3^*(t) \end{bmatrix} = \begin{bmatrix} q_{1x}^*(4-t) & -q_{1y}^*(4-t) \\ q_{2x}^*(4-t) & -q_{2y}^*(4-t) \\ q_{3x}^*(4-t) & -q_{3y}^*(4-t) \end{bmatrix}, \quad t \in (2, 4].$$

It follows that at  $t = 4$ ,  $q_i^*(4) = q_i^*(0)$  ( $i = 1, 2, 3$ ) and the velocities satisfy  $\dot{q}_i^*(4) = \dot{q}_i^*(0)$  ( $i = 1, 2, 3$ ). Hence,  $q^*(t)$  can be extended to a periodic solution by (58) and (59), which has a period 4. The proof is complete.  $\square$

At the end of this section, we show that if the minimizer  $\mathcal{P}$  has no collision, then the periodic orbit generated by  $\mathcal{P}$  has  $D_2$  symmetry.

**Lemma 6.3.** *For any  $t \in \mathbf{R}$ ,*

$$(60) \quad q_i^*(t) = R_x q_i^*(-t), \quad (i = 1, 2, 3),$$

$$(61) \quad q_1^*(t+2) = R_x R_y q_1^*(t), \quad q_2^*(t+2) = R_x R_y q_3^*(t), \quad q_3^*(t+2) = R_x R_y q_2^*(t),$$

where  $R_x$  and  $R_y$  is defined as follows

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

*Proof.* Actually, by the extension formula (59), it is clear that

$$q_i^*(t) = R_x q_i^*(4-t) = R_x q_i^*(-t), \quad (i = 1, 2, 3).$$

By the extension formula (58), we have

$$q_1^*(2-t) = R_y q_1^*(t), \quad q_2^*(2-t) = R_y q_3^*(t), \quad q_3^*(2-t) = R_y q_2^*(t).$$

It follows that

$$q_1^*(2+t) = R_x R_y q_1^*(t), \quad q_2^*(2-t) = R_x R_y q_3^*(t), \quad q_3^*(2-t) = R_x R_y q_2^*(t).$$

The proof is complete.  $\square$

## Acknowledgements

We would like to thank A. Chenciner for valuable discussions on the existence of the Broucke-Héon solutions. Zhifu Xie is also grateful to Shanzhong Sun for valuable conversations and his hospitality during Zhifu's visit to the Capital Normal University of China. Zhifu Xie gratefully acknowledges the support from NSF grant HRD-1409939. Duokui Yan is partially supported by NNSFC (No. 11432001) and the Fundamental Research Funds of the Central Universities.

## References

- [1] A. Chenciner, R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* 152 (2000) 881–901.
- [2] A. Chenciner, Action minimizing solutions in the Newtonian n-body problem: from homology to symmetry, *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, Higher Ed. Press, Beijing, 279–294, 2002.
- [3] R. Broucke, On relative periodic solutions of the planar general three-body problem, *Celest. Mech.* 12 (1975) 439–462.
- [4] K. Chen, Removing Collision Singularities from Action Minimizers for the N-Body Problem with Free Boundaries, *Arch. Rational Mech. Anal.* 181 (2006) 311–331.
- [5] K. Chen, Y. Lin, On action-minimizing retrograde and prograde orbits of the three-body problem, *Commun. Math. Phys.* 291 (2009) 403–441.
- [6] K. Chen, T. Ouyang, Z. Xia, Action-minimizing periodic and quasi-periodic solutions in the n-body problem, *Math. Res. Lett.* 19 (2012) 483–497.
- [7] D. Ferrario, S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invent. math.* 155 (2004) 305–362.
- [8] M. Hénon, Families of periodic orbits in the planar three-body problem, *Celest. Mech.* 10 (1974) 375–388.
- [9] M. Hénon, A family of periodic solutions of the planar three-body problem, and their stability, *Celest. Mech.* 13 (1976) 267–285.
- [10] W. B. Gordon, A minimizing property of Keplerian orbits, *Amer. J. Math.* 99 (1977) 961–971.
- [11] Y. Long, S. Zhang, Geometric characterizations for variational minimization solutions of the 3-body problem, *Acta Math. Sin. (Engl. Ser.)* 16 (2000) 579–592.
- [12] C. Marchal, How the method of minimization of action avoids singularities, *Celest. Mech. Dyn. Astro.* 83 (2002) 325–353.
- [13] R. Moeckel, A topological existence proof for the Schubart orbits in the collinear three-body problem. *Disc. Cont. Dyn. Syst. Ser. B* 10 (2008) 609–620.
- [14] T. Ouyang, Z. Xie, Star pentagon and many stable choreographic solutions of the Newtonian 4-body problem, *Phys. D* 307 (2015) 61–76.
- [15] T. Ouyang, Z. Xie, A continuum of periodic solutions to the planar four-body problem with various choices of masses, submitted.
- [16] D. Saari, The manifold structure for collision and for hyperbolic-parabolic orbits in the N-body problem, *J. Diff. Eqn.* 55 (1984) 300–329.

- [17] C. Simó, E. Lacomba, Regularization of simultaneous binary collisions in the  $N$ -body problem, *J. Diff. Eqn.* 55 (1992) 241–259.
- [18] J. Schubart, Numerische aufsuchung periodischer lösungen im dreikörperproblem, *Astr. Nachr.* 283 (1956) 17–22.
- [19] H.J. Sperling, On the real singularities of the  $N$ -body problem, *J. Reine Angew. Math.* 245 (1970) 15–40.
- [20] K.F. Sundman, Mémoire sur le problèdes trois corps. *Acta Math.* 36 (1913) 105–179.
- [21] S. Terracini, A. Venturelli, Symmetric trajectories for the  $2N$ -body problem with equal masses, *Arch. Rational Mech. Anal.* 184 (2007) 465–493.
- [22] A. Venturelli, A variational proof of the existence of von Schubart’s orbit, *Disc. Cont. Dyn. Syst. Ser. B* 10 (2008) 699–717.
- [23] E. Mateus, A. Venturelli, C. Vidal, Quasiperiodic collision solutions in the spatial isosceles three-body problem with rotating axis of symmetry, *Arch. Ration. Mech. Anal.* 210 (2013) 165–176.
- [24] T. Ouyang, D. Yan, Simultaneous binary collisions in the equal-mass collinear four-body problem, *Electron. J. Differential Equations* 80 (2015) 1–34.
- [25] D. Yan, T. Ouyang, New phenomena in the spatial isosceles three-body problem, *Int. J. Bifurcation Chaos* 25 (2015) 1550116.